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XIX. *Thermal Stresses in Cylindrical Pipes.*

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1. *Introduction.*

THE present paper deals with steady-state temperature stresses in a cylindrical tube. The author's interest in this problem was aroused by the question of stresses in tubes of mercury boilers where the tubes are exposed to the flame and receive radiation on one side only.

To simplify the problem it is assumed that the temperature is independent of distance along the tube axis, thus rendering it two-dimensional; that the coefficient of thermal conductivity and the coefficient of thermal expansion k, as well as the elastic constants (Young's modulus E and Poisson's ratio σ), are constant. Under these assumptions a *complete analytical solution* of the problem is obtained. What makes this possible is, very largely, the fact that the temperature itself is a harmonic function.

If the tube is not restricted at its ends, it will expand to a length determined by the mean temperature, and bow or curve in an arc of a circle, the curvature being determined by the moment of the temperature distributions. The above behaviour of the tube does not set

* Communicated by the Author.

up any stresses, and thus relieves in part the stress-producing tendencies of the temperature distribution. What is not relieved in this way will be denoted by T′ and referred to as the " reduced " temperature. In case the ends are not free it is still advisable to imagine the ends free and use the reduced temperature T′. It is necessary to add, however, further stresses due to a proper tension (or compression) and bending moments applied over the ends so as to bring them back to their restrained position. These are determined from familiar beam (or strut and tie) theory. In particular, the tube is liable to buckle if prevented from expanding. In the following, then, it will be assumed that the ends are free.

Due to the reduced temperature T′, it turns out there is no displacement in the direction of the tube or cylinder axis. Only the displacement components u, v normal to the axis are to be considered, and these are independent of the distance along the axis. It is shown that u, v are conjugate harmonic functions so that $u+iv$ is an analytic function of $x+iy$. A solution of the thermo-elastic equations is given, in fact, by

$$u+iv=\int \phi(x+iy)d(x+iy), \quad . \quad . \quad . \quad . \quad (1)$$

where ϕ is the analytic function whose real part is the harmonic function $k\mathrm{T}'$:

$$\mathrm{R}[\phi(x+iy)]=k\mathrm{T}'. \quad . \quad . \quad . \quad . \quad (2)$$

For this solution all stress components vanish except for Z_z.

For a simply connected (solid) cylinder the above solution leaves nothing to be desired. The displacements are given by (1), while the only non-vanishing stress component is the axial tensile component Z_z, given by

$$\mathrm{Z}_z=\mathrm{E}k\mathrm{T}'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For multiply-connected regions—that is, for a cylinder with one or more cylindrical holes—the integral in (1) may not evaluate to zero for a path enclosing a hole, so that the above u, v will become multiple-valued. The true state of stress and displacement may then be thought of as consisting of two parts : one given by (1), and having no stresses X_x, X_y, Y_y, and the other having no temperature inequalities and no boundary tractions, but

eliminating the multiple-valuedness of the displacements of the particular solution (1). This follows from the fact that the elasticity equations are linear and homogeneous in the stress and displacement components, except for the temperature terms. Now the general solution of a linear non-homogeneous system consists of any particular solution plus a general solution of the corresponding homogeneous system—in this case, of the elasticity equations without temperature terms.

The multiple-valuedness of (1), it will be shown, is of such a nature that in going once around a hole the displacement is increased by a rigid displacement (*i. e.*, a rotation and a translation).

The correcting state of stress and the displacements are therefore in the nature of what is known as " dislocations," that is, they represent states of stress with zero tractions over the boundary and with displacements that are multiple-valued, the various determinations differing from each other by a translation and a rigid rotation. More particularly, let cuts be made from the holes to each other or to the outer boundary in such a way as to render the region simply connected, or deformable into a circle without further cutting or joining of boundary points. The displacements for the simply-connected region are single-valued, but their values on opposite sides of the cut differ by a translation and rotation, and a similar difference exists between the continuations of the values from either side of the cut across that cut into overlapping regions. The theory of " dislocations " is due to Michell and Volterra (Love, ' Mathematical Theory of Elasticity,' 4th ed., Appendix to Chapters VIII. and IX.). A direct proof of their applicability in this case which does not involve the theory of dislocations is obtained in the following, and the Airy's function corresponding to these dislocations for the region between concentric circles is obtained. The convenient complex form is employed for the Airy's function :

$$\mathrm{F}=\mathrm{R}[f+(x-iy)g], \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where f, g are analytic functions of $x+iy$.

After this investigation was completed, our attention was called to a paper by M. A. Biot (' Journal of Applied Mechanics,' A. 41, vol. 2, no. ii. June 1935) on stresses due to gravity and temperature loading in which the

reduction to dislocations is indicated. The presence of *axial* stresses even for simply connected regions is not, however, pointed out by Biot.

2. *List of Symbols.*

x, y, z=Cartesian coordinates, the z-axis pointing along the tube-axis,

r, θ, z=Cylindrical coordinates.

$\mathbf{r}$=Displacement vector.

u, v, w=Displacement components along the x-, y-, z-axes.

T=Temperature.

T′=Reduced temperature $= T-(T_m+Cx+C'y)$, where T_m, C, C′ are proper constants. The temperature T′ possesses a vanishing mean and vanishing moments about the axes.

k=Coefficient of linear expansion.

E=Young's modulus.

σ=Poisson's ratio.

$X_x, X_y, \ldots$=Stress components.

$e_{xx}, e_{xy}, \ldots$=Strain components.

$\theta=e_{xx}+e_{yy}+e_{zz}$=Divergence.

$Z=x+iy$.

$\overline{Z}=x-iy$.

3. *The Thermo-elastic Equations.*

The Reduced Temperature.

Adding the heat expansions to the elastic Hooke-Law expansions, the stress-strain equations become

$$\left.\begin{aligned} e_{xx} &= \frac{1}{E}[(1+\sigma)X_x-\sigma(X_x+Y_y+Z_z)]+kT, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ e_{xy} &= \frac{2(1+\sigma)}{E}X_y, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad (5)$$

where

$$e_{xx}=\frac{\partial u}{\partial x}, \qquad e_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},$$

and the indicated equations are obtained by permuting the x, y, z; u, v, w components.

Adding the first three equations, there results

$$\theta=e_{xx}+e_{yy}+e_{zz}=\frac{1-2\sigma}{E}(X_x+Y_y+Z_z)+3k\mathrm{T}, \quad . \quad (6)$$

whence

$$(X_x+Y_y+Z_z)=\frac{E}{1-2\sigma}\theta-\frac{3k\mathrm{T}E}{1-2\sigma} \quad . \quad . \quad . \quad . \quad (7)$$

Substituting in the strain-stress equations and solving for the stresses,

$$\left.\begin{aligned} X_x&=\frac{E}{1+\sigma}\left[e_{xx}+\frac{\sigma}{1-2\sigma}\theta-\frac{1+\sigma}{1-2\sigma}k\mathrm{T}\right], \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ X_y&=\frac{E}{2(1+\sigma)}e_{xy}, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad . \quad . \quad (8)$$

Substituting in the equations of equilibrium

$$\left.\begin{aligned} &\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=0, \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

there results

$$\left.\begin{aligned} &\nabla^2 u+\frac{1}{1-2\sigma}\frac{\partial\theta}{\partial x}-\frac{2(1+\sigma)k}{(1-2\sigma)}\frac{\partial \mathrm{T}}{\partial x}=0, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad . \quad . \quad (10)$$

or in vector form,

$$\nabla^2\mathbf{r}+\frac{1}{1-2\sigma}\nabla[\epsilon-2k(1+\sigma)\mathrm{T}]=0. \quad . \quad . \quad (11)$$

As a first example consider a uniform temperature rise. This will result in a uniform elongation along the cylinder axis. In treating an arbitrary temperature

distribution $T(x, y)$ we shall first allow for a uniform expansion along the axis (as well as at right angles to it) corresponding to the *mean* temperature T_m over the cylinder cross-section.

Placing the x, y-axes along the principal axes of inertia through the centre of gravity of the cross-section, consider next temperature distributions

$$T=Cx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where C is a constant. The strain-stress (Hooke's Law) equations (5) are satisfied by having all the stress components vanish, hence by

$$e_{xx}=e_{yy}=e_{zz}=kT=(kC)x \quad . \quad . \quad . \quad . \quad (13)$$
$$e_{xy}=e_{xz}=e_{yz}=0.$$

The axial fibre elongation

$$e_{zz}=\frac{\partial w}{\partial z}=kCx$$

corresponds to a bending or bowing of the cylinder about the y-axis, so that the z-axis and lines parallel to it acquire a curvature

$$kC. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Similarly, the strain e_{yy} results in a change of cross-section which bends the y-axis into an arc of a circle of the same curvature as the cylinder axis (the z-axis). Thus, planes parallel to the y–z plane are curved into spherical surfaces of radius $1/kC$. The effect of the strain e_{xx} is to spread these spherical surfaces apart on the side of positive x and to condense them on the opposite side.

While the nature of the above solution is obvious physically, to complete it mathematically one has to find displacement components u, v, w corresponding to the strains (13). These are given by

$$\left.\begin{aligned} u&=\frac{kC}{2}(x^2-y^2-z^2), \\ v&=kCxy, \quad w=kCxz. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15)$$

Similarly a temperature distribution

$$T=C'y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

results in a stress-free displacement

$$u = kC'xy, \quad V = \frac{kC'}{2}(y^2 - x^2 - z^2), \quad w = kC'yz. \quad (17)$$

Consider now an arbitrary distribution $T = T(x, y)$. Put

$$T = T_m + Cx + C'y + T', \quad . \quad . \quad . \quad (18)$$

where T' has a vanishing mean and a vanishing moment about both the x- and y-axis. The values T_m, C, C′ are found by multiplying both sides of (18) by 1, x, y, and integrating over the cross-section area A. There results

$$\left.\begin{aligned} \int T dA &= T_m A, \\ \int Tx dA &= CI_y, \\ \int Ty dA &= C'I_x, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

where I_x, I_y are the moments of inertia of the cross-section about the y- and x-axes. The displacements and stresses due to the first three terms are found as above outlined. In the following we shall, therefore, consider only the stresses and displacements due to the reduced temperature T′.

4. *A Particular Solution.*

We now assume that, due to the reduced temperature T′, there are no displacements parallel to the x-axis,

$$w = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

while u, v depend only on x and y. The elasticity equations (10) then reduce to two in number with two-dimensional ∇, ∇^2. When these are solved and the stress components X_x, X_y, Y_y are determined from the proper equations (8), the further stress components are found from the third equation (5) thus :

$$e_{zz} = 0 = \frac{1}{E}[Z_z(1+\sigma) - (X_x + Y_y + Z_z)] + kT',$$

whence

$$Z_z = \sigma(X_x + Y_y) - EkT' ; \quad . \quad . \quad . \quad . \quad (21)$$

the stress components X_z, Y_z vanish identically.

We shall examine first if it is possible to have temperature distributions that give rise to no stresses X_x, X_y, Y_y.

If we equate these stress components to zero, equations (5) yield

$$e_{xx}=k\mathrm{T}',$$
$$e_{yy}=k\mathrm{T}',$$
$$e_{xy}=0,$$

or

$$\frac{\partial u}{\partial x}=k\mathrm{T}', \quad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0, \quad \frac{\partial v}{\partial y}=k\mathrm{T}'. \quad . \quad (22)$$

Hence

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

These are the Cauchy-Riemann equations stating that $u+iv$ is an analytic function of $x+iy$. In particular, u and v are both harmonic, and so is $\frac{\partial u}{\partial x}=k\mathrm{T}'$. This shows that for non-steady state temperature distributions the above solution will in general not apply. To complete the above solution let $u+iv$ be given by (1), where ϕ is given by (2). From the definition of an analytic function it follows that

$$\frac{d(u+iv)}{d(x+iy)}=\phi(x+iy)=\frac{\partial(u+iv)}{\partial x}=\frac{\partial(u+iv)}{i\,\partial y}$$
$$=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}.$$

Equating the real parts of the various terms there result equations (22).

5. *The Complementary Solution.*

As explained in section 1, the special solution just found may not suffice for multiply-connected regions. In a region between two concentric circles the harmonic function $k\mathrm{T}'$ may be expanded in the following series :

$$k\mathrm{T}'=\mathrm{A}\ln r+\sum_{n=-\infty}^{n=+\infty}(\mathrm{A}_n\cos n\theta+\mathrm{B}_n\sin n\theta)r^n. \quad (23)$$

$$=\mathrm{R}[\mathrm{A}\ln \mathrm{Z}+\Sigma(\mathrm{A}_n-i\mathrm{B}_n)\mathrm{Z}^n], \quad . \quad . \quad . \quad . \quad (24)$$

where $\mathrm{Z}=x+iy$.

Comparing with (2) we see that the bracket in (24) may be equated to $\phi(Z)$, whereupon (1) yields

$$u+iv=\int\phi(Z)dZ=A_0(Z\ln Z-Z)+(A_{-1}-iB_{-1})\ln Z + \sum{}'\frac{(A_n-iB_n)}{n+1}Z^{n+1}, \quad (25)$$

where the prime in the summation indicates that in the summation the value $n=-1$ is excluded.

The only multiple-valued terms are those preceding the summation, and these increase by

$$2\pi i(AZ+A_{-1}-iB_{-1}) \quad . \quad . \quad . \quad . \quad (26)$$

upon going once around the origin. But such an increase in $u+iv$ represents a translation $(2\pi B_{-1},\ 2\pi A_{-1})$ and a rotation through an angle $2\pi A_0$ about the origin (the usual restriction to small displacements being understood). The correcting solution must therefore be a dislocation which counteracts the multiple-valuedness (26) and is due only to the terms $A\ln r$, $(A_{-1}\cos\theta+B_{-1}\sin\theta)r^{-1}$ in (23).

For general ring-shaped regions, that is—regions between two closed curves, one of which encloses the other—the expansion (24) is not valid, yet the same results still hold. The proof may be based upon the fact that a finite sum of the form (23) can be made to approximate to a single-valued harmonic function in an arbitrary ring-shaped region, the coefficients as well as the number of terms changing with the approximation desired. (This result is due to Bergmann and Walsh.)

The complementary dislocation will be expressed in terms of its Airy's function F, for which the elegant form (4) will be employed. Introducing the variables $Z=x+iy$, $\bar{Z}=x-iy$, we have

$$F=R(f+\bar{Z}g)=\frac{f+\bar{Z}g+\bar{f}+Z\bar{g}}{2}, \quad . \quad . \quad . \quad (27)$$

where f, g are analytic in Z, and bars denote conjugates (thus $\bar{g}=\overline{g(\bar{Z})}$).

At a free boundary the condition,

$$2\frac{\partial F}{\partial\bar{Z}}=g+\bar{f}'+\bar{Z}\bar{g}'=\text{const.} \quad . \quad . \quad . \quad . \quad (28)$$

holds.

The displacements are given by

$$2G(u+iv)=(3-4\sigma)g-\bar{f}'-Z\bar{g}', \quad . \quad . \quad . \quad (29)$$

where $G=E/2(1+\sigma)$ is the shear modulus. For derivation of (28) and (29) see Appendix.

We shall consider the complementary solution corresponding to each term of (26) separately, first taking account of $2\pi iAZ$. We choose an Airy's function depending on r only :

$$\begin{aligned} F&=C_1r^2+C_2\ln r+C_3r^2\ln r \qquad (C_1, C_2, C_3 \textit{ real}). \quad (30)\\ &=R[C_2\ln Z+(r^2=Z\bar{Z})(C_1+C_3\ln Z)]\\ &=R[C_2\ln Z+\bar{Z}(C_1Z+C_3Z\ln Z)], \end{aligned}$$

yielding for f, g in (27)

$$f=C_2\ln Z, \quad g=C_1Z+C_3Z\ln Z. \quad . \quad . \quad . \quad (31)$$

From equation (29) we now obtain

$$\begin{aligned} 2G(u+iv)=(3-4\sigma)[C_1Z+C_3\ln Z]. \\ -Z[(C_1+C_3(\ln\bar{Z}+1)]-\frac{C_2}{\bar{Z}}. \end{aligned}$$

When the origin is encircled $(u+iv)$ is increased by

$$2\pi i\frac{C_3(4-4\sigma)Z}{2G}=2\pi i\frac{2(1-\sigma)Z}{G}C_3.$$

Hence we put

$$\frac{2(1-\sigma)}{G}C_3+A=0, \quad C_3=-\frac{AG}{2(1-\sigma)}. \quad . \quad (32)$$

Substituting (31) in (28), we find that

$$C_1Z+C_3Z\ln Z+\frac{C_2}{\bar{Z}}+Z[C_1+C_3(\ln\bar{Z}+1)] \quad . \quad (33)$$

must be constant along the boundaries $r=a$, $r=b$. But (33) is equal to

$$\begin{aligned} e^{i\theta}[r(2C_1+C_3)+\frac{C_2}{r}+2C_3r\ln r] \\ =e^{i\theta}[C_3(r+2r\ln r)+C_1(2r)+\frac{C_2}{r}). \end{aligned}$$

This can be constant only if the bracket reduces to zero; hence, if

$$\left.\begin{aligned} C_1(2a)+\frac{C_2}{a} &= -C_3 a(1+2\ln a),\\ C_1(2b)+\frac{C_2}{b} &= -C_3 b(1+2\ln b). \end{aligned}\right\} \quad . \quad . \quad (34)$$

These equations determine C_1, C_2.

The state of stress just found corresponds to the removal of a sector-shaped slice (of angle $2\pi A$) followed by cementing the edges.

Consider next the term $2\pi i A_{-1}$ in (26). We shall set

$$\left.\begin{aligned} g &= C_4 Z^2 + C_5 \ln Z,\\ f' &= \frac{C_6}{Z^2} + C_7 \ln Z, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where $C_4 \ldots$ are again real.

We obtain from (29)

$$2G(u+iv) = (3-4\sigma)(C_4 Z^2 + C_5 \ln Z) - \frac{C_6}{\bar{Z}} - C_7 \ln \bar{Z} - Z\left(2C_4 Z - \frac{C_5}{\bar{Z}}\right).$$

When Z encircles the origin $(u+iv)$ increases by

$$2\pi i \frac{(3-4\sigma)C_5 + C_7}{2G}.$$

Equating this to $-2\pi i A_{-1}$ yields

$$(3-4\sigma)C_5 + C_7 = -2G\,A_{-1}. \quad . \quad . \quad . \quad (36)$$

Turning to (28), we find for the middle member $g+\bar{f}'+Z\bar{g}'$

$$C_4 Z^2 + C_5 \ln Z + Z\left(2C_4 \bar{Z} + \frac{C_5}{Z}\right) + \frac{C_6}{\bar{Z}^2} + C_7 \ln \bar{Z}$$

$$= e^{2i\theta} C_4 r^2 + C_5 + \frac{C_6}{r^2} + 2C_4 r^2 + i\theta(C_5 - C_7) + (C_5 + C_7)\ln r.$$

To keep this constant over $r=a$, $r=b$ we must first have

$$C_5 - C_7 = 0, \quad C_5 = C_7; \quad . \quad . \quad . \quad . \quad (37)$$

then

$$\left.\begin{aligned} C_4a^2+\frac{C_6}{a^2}&=-C_5,\\ C_4b^2+\frac{C_6}{b^2}&=-C_5. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Equations (36), (37), and (38) determine the constants C_4–C_7 completely.

The dislocation just found corresponds to the removal near the positive x-axis of material of width $(2\pi A_{-1})$ between lines (planes) parallel to the x-axis.

Summarizing, we see that the complementary state of stress which annuls the multiple-valuedness of the first two terms in (26) is described by the Airy's function (27) with

$$\left.\begin{aligned} f&=C_2 \ln Z-\frac{C_6}{Z}+C_7(Z \ln Z-Z),\\ g&=C_1Z+C_3Z \ln Z+C_4Z^2+C_5 \ln Z, \end{aligned}\right\} \quad . \quad . \quad (39)$$

where C_1 are determined by (32), (34), (36)–(38).

As regards the term $2\pi i(-iB_{-1})$ in equation (26), we see that it can be taken care of by means of a rotation of the previous solution through 90° and a multiplication by B_{-1}/A_{-1}. Incidentally, this term may be made to disappear altogether by carrying out proper rotation of axes.

The stress components can be found, as usual, by differentiating F twice ; thus,

$$(X_x, X_y, Y_y)=\left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x\,\partial y}, \frac{\partial^2}{\partial x^2}\right)F. \quad . \quad (40)$$

In polar coordinates the stress components are given by

$$(p_{rr}, p_{r\theta}, p_{\theta\theta})=\left(\frac{\partial^2}{r^2\,\partial\theta^2}+\frac{1}{r}\frac{\partial}{r\,\partial r}, \frac{1}{r^2}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta}, \frac{\partial^2}{\partial r^2}\right)F.$$

$$. \quad . \quad . \quad (41)$$

Applying equation (40) to (27), we obtain for $p_{\theta\theta}$

$$p_{\theta\theta}=\frac{\partial^2 F}{\partial r^2}=R[e^{2i\theta}(f''+\bar{Z}g'')+2g']. \quad . \quad . \quad (42)$$

Carrying out the differentiations and simplifying, there results for vanishing B_{-1}

$$p_{\theta\theta}=\left(-\frac{C_2}{r^2}+2C_1+3C_3+2C_3\ln r\right)$$
$$+\left(\cos\theta-\frac{2C_6}{r^3}+\frac{C_5+C_7}{r}+6\,C_4 r\right). \quad (43)$$

The maximum of this is taken on at $\theta=(0 \text{ or } \pi)$ and $r=(a \text{ or } b)$.

Since the special solution (1) involved no stresses p_{rr}, $p_{r\theta}$, $p_{\theta\theta}$, the stress $p_{\theta\theta}$ in the complete solution is still given by (43). The stress $p_{zz}=Z_z$ is given by (21). At the boundary where its maximum occurs (21) reduces to

$$Z_z=\sigma p_{\theta\theta}-E k T'. \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Another way of deriving the stresses from the Airy's function is by utilizing

$$\frac{2}{i}\frac{\partial F}{\partial \bar{Z}}=-i(g+Z\bar{g}'+\bar{f}'). \quad . \quad . \quad . \quad . \quad (45)$$

By differentiating this in any direction the traction across a curve in that direction will result (both in magnitude and direction). (For proof see Appendix.)

Appendix.

Airy's function satisfies the repeated Laplace equation,

$$\nabla^4 F=0.$$

By changing to Z, $\bar{Z}$ as variables, one transforms ∇^2 to $4\partial^2/\partial Z\,\partial\bar{Z}$, and the above differential equation to

$$\frac{\partial^2 F}{\partial Z^2\,\partial\bar{Z}^2}=0.$$

Successive integration yields

$$F=f_1(Z)+\bar{Z}g_1(Z)+f_2(\bar{Z})+Zg_2(Z).$$

Taking account of the fact that F is real for real x, y, one proves that $f_2(\bar{Z})=\overline{f_1(Z)}$, $g_2(\bar{Z})=\overline{g_1(Z)}$ and obtains equation (25).

To derive (26), recall that $\frac{\partial F}{\partial y}\Big]_{P_1}^{P_2}$ gives the traction component in the x-direction across any cut made from P_1 to P_2. Similarly $-\frac{\partial F}{\partial x}\Big]_{P_1}^{P_2}$ gives the analogous traction component in the y-direction. Hence

$$\frac{\partial F}{\partial y} - i\frac{\partial F}{\partial x}\Big]_{P_1}^{P_2}$$

yields that traction considered as a complex number. Changing variables once more from x, y to Z, $\bar{Z}$, one shows that $\partial F/\partial \bar{Z} = \frac{i}{2}\left(\frac{\partial F}{\partial y} - i\frac{\partial F}{\partial x}\right)$.

Over a free boundary the traction vanishes; hence $\partial F/\partial \bar{Z}$ is constant, hence (26). The statement made in connection with equation (45) also readily follows from the above.

The form (27) is due, we believe, to Goursat, while (28) is to be credited to N. Muschelisvili *, through it was obtained independently by the author in a paper reported to the American Mathematical Society in April 1933 (but not yet published). We review its derivation once more.

The displacement in terms of Airy's function is found as follows :—

Let s be the harmonic function

$$s = \nabla^2 F$$

and t its conjugate harmonic, and let

$$S + iT = \int (s + it) d(x + iy),$$

then (see E. Trefftz in Geiger and Scheel's ' Handbuch der Physik,' vol. vi. p. 110)

$$(2G)(u, v) = (1 - \sigma)(S, T) - \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)F.$$

In terms of the functions f, g in (27) this takes on the form

$$s = 4\frac{\partial^2 F}{\partial Z \partial \bar{Z}} = 2(g' + \bar{g}'),$$

$$s + it = 4g', \quad S + iT = 4g$$

* *Zeitschrift für angewandte Mathematik und Mechanik*, vol. xiii. pp. 264–265 (1933).

$$(2G)(u+iv)=4(1-\sigma)g-\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)F$$
$$=4(1-\sigma)g-2\frac{\partial F}{\partial \bar{Z}},$$

or $$(2G)(u+iv)=(3-4\sigma)g-f'-Zg\,.$$

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XX. *Some Series and Integrals for the Associated Legendre Functions of the First Kind, regarded as Functions of their Orders.* *By* T. M. MacRobert, *The University, Glasgow* *.

§ 1. *Preliminary Formulæ.*

In a former paper (Phil. Mag. xxi. p. 702 (1936)) the formula

$$2\pi\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\mathrm{T}_n^{-m}(\cos\theta)=\int_{-\pi}^{\pi}\frac{e^{im\phi}d\phi}{(\cos\theta+i\sin\theta\,\mathrm{s\,n}\,\phi)^{n+1}}$$
$$-2\sin m\pi\int_0^{\infty}\frac{e^{-mu}du}{(\cos\theta+\sin\theta\sinh u)^{n+1}},\quad\ldots\quad(1)$$

where $0<\theta<\frac{1}{2}\pi$, $\mathrm{R}(n+m+1)>0$, was obtained. It resembles Whipple's formula (Proc. Lond. Math. Soc. xvi. p. 109 (1917))

$$\pi\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\mathrm{P}_n^{-m}(z)=\int_0^{\pi}\frac{\cos m\theta\,d\theta}{\{z-\sqrt{(z^2-1)}\cos\theta\}^{n+1}}$$
$$-\sin m\pi\int_0^{\infty}\frac{e^{-mu}du}{\{z+\sqrt{(z^2-1)}\cosh u\}^{n+1}},\quad\ldots\quad(2)$$

where $\mathrm{R}(z)>0$, $z\neq 1$, $\mathrm{R}(n+m+1)>0$; and, indeed, the connexion between them can be established as follows:

Formula (2) is equivalent to the formula

$$\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\mathrm{P}_n^{-m}(z)=\frac{1}{2\pi i}\int_C\frac{\zeta^{-m-1}d\zeta}{\{z-\frac{1}{2}\sqrt{(z^2-1)(\zeta+1/\zeta)}\}^{n+1}},$$
$$\ldots\quad(3)$$

* Communicated by the Author.

where the contour C starts from $-\infty$ on the real axis, passes round the origin in the positive direction and returns to $-\infty$ on the real axis, amp. $\zeta=-\pi$ initially,

$$\mathrm{R}(z)>0, \quad z\neq 1, \quad \mathrm{R}\{z-\tfrac{1}{2}\sqrt{(z^2-1)}(\zeta+1/\zeta)\}>0,$$

and $$\mathrm{R}(n+m+1)>0.$$

If $0<x<1$, and x is reached by passing z round $z=1$ in the positive direction,

$$\mathrm{T}_n^{-m}(x)=e^{-\frac{1}{2}m\pi i}\,\mathrm{P}_n^{-m}(x)$$
$$=\frac{\Gamma(n+1)}{\Gamma(n+m+1)}\,\frac{e^{-\frac{1}{2}m\pi i}}{2\pi i}\int_{\mathrm{C}}\frac{\zeta^{-m-1}d\zeta}{\{x-\frac{1}{2}i\sqrt{(1-x^2)}(\zeta+1/\zeta)\}^{n+1}}.$$

Now turn C through an angle $-\frac{1}{4}\pi$, put $\zeta=\mathrm{Z}/i$, so that the contour in the Z-plane is the contour in the ζ-plane turned through an angle $+\frac{1}{2}\pi$, and next turn this new contour through an angle $-\frac{1}{4}\pi$. Then, writing ζ for Z, we have

$$\mathrm{T}_n^{-m}(x)=\frac{\Gamma(n+1)}{\Gamma(n+m+1)}\,\frac{1}{2\pi i}\int_{\mathrm{C}}\frac{\zeta^{-m-1}d\zeta}{\{x-\frac{1}{2}\sqrt{(1-x^2)}(\zeta-1/\zeta)\}^{n+1}}, \qquad (4)$$

where $0<x<1$, $\mathrm{R}\{x-\frac{1}{2}\sqrt{(1-x^2)}(\zeta-1/\zeta)\}>0$,

and $$\mathrm{R}(n+m+1)>0.$$

Formulæ (4) and (1) are equivalent.

Formula (3) may be obtained by substituting for $\mathrm{I}_m\{(\lambda\sqrt{z^2-1})\}$ in

$$\mathrm{P}_n^{-m}(z)=\frac{1}{\Gamma(n+m+1)}\int_0^\infty e^{-\lambda z}\,\mathrm{I}_m\{\lambda\sqrt{(z^2-1)}\}\lambda^n d\lambda, \qquad (5)$$

where $\mathrm{R}(z)>0$, $z\neq 1$, $\mathrm{R}(n+m+1)>0$, using the formula

$$\mathrm{I}_m(z)=\frac{1}{2\pi i}\int_{\mathrm{C}}e^{\frac{1}{2}z(\zeta+1/\zeta)}\zeta^{-m-1}d\zeta, \quad \mathrm{R}(z)>0,$$

and then changing the order of integration.

§ 2. *Integrals and Series.*

The following formulæ may be derived from formula (1):

$$\frac{2\pi}{\Gamma(\lambda+1)}\int_0^\infty \Gamma(\lambda+\mu+1)\cos\mu\phi\;\mathrm{T}_\lambda^{-\mu}(\cos\theta)\,d\mu$$
$$=\tfrac{1}{2}\pi\{\mathrm{F}(\lambda,\theta,\phi)+\mathrm{F}(\lambda,\theta,-\phi)\}$$

$$+\frac{i}{2}\mathrm{P}\int_{-\pi}\frac{1}{\psi+\phi}\{\mathrm{F}(\lambda,\theta,\psi)-\mathrm{F}(\lambda,\theta,-\psi)\}d\psi$$

$$-\int_0^\infty\{\mathrm{G}(u,\phi)+\mathrm{G}(u,-\phi)\}\frac{du}{(\cos\theta+\sin\theta\sinh u)^{\lambda+1}},$$

. . . (6)

where $0\leqq\theta<\frac{1}{2}\pi$, $\mathrm{R}(\lambda+1)>0$, $-\pi<\phi<\pi$,

$$\mathrm{F}(\lambda,\theta,\phi)\equiv\frac{1}{(\cos\theta+i\sin\theta\sin\phi)^{\lambda+1}}$$

and $$\mathrm{G}(u,\phi)\equiv\frac{\pi+\phi}{u^2+(\pi+\phi)^2};$$

$$\frac{2\pi}{\Gamma(\lambda+1)}\int_0^\infty\Gamma(\lambda+\mu+1)\sin\mu\phi\,\mathrm{T}_\lambda^{-\mu}(\cos\theta)\,d\mu$$

$$=\tfrac{1}{2}\pi i\{\mathrm{F}(\lambda,\theta,\phi)-\mathrm{F}(\lambda,\theta,-\phi)\}$$

$$+\tfrac{1}{2}\mathrm{P}\int_{-\pi}^{\pi}\frac{1}{\psi+\phi}\{\mathrm{F}(\lambda,\theta,\psi)+\mathrm{F}(\lambda,\theta,-\psi)\}d\psi$$

$$+\int_0^\infty\{\mathrm{H}(u,\phi)-\mathrm{H}(u,-\phi)\}\frac{du}{(\cos\theta+\sin\theta\sinh u)^{\lambda+1}},$$

. . . (7)

where $0\leqq\theta<\frac{1}{2}\pi$, $\mathrm{R}(\lambda+1)>0$, $-\pi<\phi<\pi$,

and $$\mathrm{H}(u,\phi)\equiv\frac{u}{u^2+(\pi+\phi)^2};$$

$$\frac{2\pi}{\Gamma(n+1)}\sum_{p=0}^{\infty}\Gamma(n+p+\alpha+1)\cos(p+\alpha)\phi\,\mathrm{T}_n^{-p-\alpha}(\cos\theta)$$

$$=\tfrac{1}{2}\pi\{\mathrm{F}(n,\theta,\phi)+\mathrm{F}(n,\theta,-\phi)\}$$

$$+\tfrac{1}{2}i\mathrm{P}\int_{-\pi}^{\pi}\frac{\cos\{(\alpha-\frac{1}{2})(\psi-\phi)\}}{2\sin\frac{1}{2}(\psi-\phi)}\{\mathrm{F}(n,\theta,\psi)-\mathrm{F}(n,\theta,-\psi)\}d\psi$$

$$-\tfrac{1}{2}\int_{-\pi}^{\pi}\frac{\sin\{(\alpha-\frac{1}{2})(\psi-\phi)\}}{2\sin\frac{1}{2}(\psi-\phi)}\{\mathrm{F}(n,\theta,\psi)+\mathrm{F}(n,\theta,-\psi)\}d\psi$$

$$-2\sin\alpha\pi\int_0^\infty\frac{e^{-\alpha u}\cos\alpha\phi+e^{-(\alpha+1)u}\cos(\alpha-1)\phi}{\{\cos\theta+\sin\theta\sinh u\}^{u+1}(1+2e^{-u}\cos\phi+e^{-2u})}du,$$

. . . (8)

where $0<\theta<\frac{1}{2}\pi$, $\mathrm{R}(n+\alpha+1)>0$, $-\pi<\phi<\pi$;

$$\frac{2\pi}{\Gamma(n+1)}\sum_{p=0}^{\infty}\Gamma(n+p+\alpha+1)\sin(p+\alpha)\phi\,\mathrm{T}_n^{-p-\alpha}(\cos\theta)$$

$$=\tfrac{1}{2}\pi i\{\mathrm{F}(n,\theta,\phi)-\mathrm{F}(n,\theta,-\phi)\}$$

$$-\tfrac{1}{2}\mathrm{P}\int_{-\pi}^{\pi}\frac{\cos\{(\alpha-\frac{1}{2})(\psi-\phi)\}}{2\sin\frac{1}{2}(\psi-\phi)}\{\mathrm{F}(n,\theta,\psi)+\mathrm{F}(n,\theta,-\psi)\}\,d\psi$$

$$-\tfrac{1}{2}i\int_{-\pi}^{\pi}\frac{\sin\{(\alpha-\frac{1}{2})(\psi-\phi)\}}{2\sin\frac{1}{2}(\psi-\phi)}\{\mathrm{F}(n,\theta,\psi)-\mathrm{F}(n,\theta,-\psi)\}\,d\psi$$

$$-2\sin\alpha\pi\int_0^{\infty}\frac{e^{-\alpha u}\sin\alpha\phi+e^{-(\alpha+1)u}\sin(\alpha-1)\phi}{\{\cos\theta+\sin\theta\sinh u\}^{n+1}(1+2e^{-u}\cos\phi+e^{-2u})}\,du, \quad \ldots \quad (9)$$

where $0<\theta<\frac{1}{2}\pi$, $\mathrm{R}(n+\alpha+1)>0$, $-\pi<\phi<\pi$.

Corresponding formulæ derived from (2) are

$$\frac{2\pi}{\Gamma(\lambda+1)}\int_0^{\infty}\Gamma(\lambda+\mu+1)\cos\mu\phi\,\mathrm{P}_{\lambda}^{-\mu}(z)\,d\mu$$

$$=\begin{cases}\mathrm{L}-\mathrm{M}, & -\pi<\phi<\pi,\\ \frac{1}{2}\mathrm{L}-\mathrm{M}, & \phi=\pm\pi,\\ -\mathrm{M}, & \phi>\pi \text{ or } \phi<-\pi,\end{cases} \quad (10)$$

where $\mathrm{R}(z)>0$, $z\neq 1$, $\mathrm{R}(\lambda+1)>0$,
and

$$\mathrm{L}\equiv\frac{\pi}{\{z-\sqrt{(z^2-1)}\cos\phi\}^{\lambda+1}},$$

$$\mathrm{M}\equiv\int_0^{\infty}\{\mathrm{G}(u,\phi)+\mathrm{G}(u,-\phi)\}\frac{du}{\{z+\sqrt{(z^2-1)}\cosh u\}^{\lambda+1}};$$

$$\frac{2\pi}{\Gamma(\lambda+1)}\int_0^{\infty}\Gamma(\lambda+\mu+1)\sin\mu\phi\,\mathrm{P}_{\lambda}^{-\mu}(z)\,d\mu$$

$$=\mathrm{P}\int_{-\pi}^{\pi}\frac{d\theta}{(\theta+\phi)\{z-\sqrt{(z^2-1)}\cos\theta\}^{\lambda+1}}$$

$$+\int_0^{\infty}\{\mathrm{H}(u,\phi)-\mathrm{H}(u,-\phi)\}\frac{du}{\{z+\sqrt{(z^2-1)}\cosh u\}^{\lambda+1}}, \quad \ldots \quad (11)$$

where $\mathrm{R}(z)>0$, $z\neq 1$, $\mathrm{R}(\lambda+1)>0$;

$$\frac{2\pi}{\Gamma(n+1)}\sum_{p=0}^{\infty}\Gamma(n+p+\alpha+1)\cos(p+\alpha)\phi\,\mathrm{P}_n^{-p-\alpha}(z)$$

$$= \frac{\pi}{\{z-\sqrt{(z^2-1)}\cos\phi\}^{n+1}}$$

$$-\int_{-\pi}^{\pi} \frac{\sin\{(\alpha-\frac{1}{2})(\psi+\phi)\}}{2\sin\frac{1}{2}(\psi+\phi)} \frac{d\psi}{\{z-\sqrt{(z^2-1)}\cos\psi\}^{n+1}}$$

$$-2\sin\alpha\pi\int_0^{\infty} \frac{e^{-\alpha u}\cos\alpha\phi+e^{-(\alpha+1)u}\cos(\alpha-1)\phi}{\{z+\sqrt{(z^2-1)}\cosh u\}^{n+1}(1+2e^{-u}\cos\phi+e^{-2u})}du,$$

. . . (12)

where $\mathrm{R}(z)>0$, $z\neq 1$, $\mathrm{R}(n+\alpha+1)>0$, $-\pi<\phi<\pi$;

$$\frac{2\pi}{\Gamma(n+1)}\sum_{p=0}^{\infty}\Gamma(n+p+\alpha+1)\sin(p+\alpha)\phi\,\mathrm{P}_n^{-p-\alpha}(z)$$

$$= \mathrm{P}\int_{-\pi}^{\pi} \frac{\cos\{(\alpha-\frac{1}{2})(\psi+\phi)\}}{2\sin\frac{1}{2}(\psi+\phi)} \frac{d\psi}{\{z-\sqrt{(z^2-1)}\cos\psi\}^{n+1}}$$

$$-2\sin\alpha\pi\int_0^{\infty} \frac{e^{-\alpha u}\sin\alpha\phi+e^{-(\alpha+1)u}\sin(\alpha-1)\phi}{\{z+\sqrt{(z^2-1)}\cosh u\}^{n+1}(1+2e^{-u}\cos\phi+e^{-2u})}du,$$

. . . (13)

where $\mathrm{R}(z)>0$, $z\neq 1$, $\mathrm{R}(n+\alpha+1)>0$, $-\pi<\phi<\pi$.

The expansion

$$\frac{1}{\{z-\sqrt{(z^2-1)}\cos\theta\}^{n+1}}$$

$$= \mathrm{P}_n(z)+2\sum_{m=1}^{\infty}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\cos m\theta\,\mathrm{P}_n^{-m}(z), \quad (14)$$

where $\mathrm{R}(z)>0$, $z\neq 1$, $\mathrm{R}(n+1)>0$, can be easily derived from (2).

To evaluate the integrals (6), (7), (10), (11), first integrate with regard to μ from 0 to μ, and then make μ tend to infinity, using the properties of the Dirichlet Integrals. For the series (8), (9), (12), (13), first sum to $(p+1)$ terms and then make p tend to infinity.

The formulæ for the function $\mathrm{P}_n^{-m}(z)$ can all be deduced, by means of Whipple's Transformation, from corresponding formulæ for the function $\mathrm{Q}_n^m(z)$. These were given by the author in a former paper (Proc. Roy. Soc. Edin. lvii. pp. 19–22 (1937)).

XXI. *On the Instability of a Spherical Shell of Fluid heated from within.* *By* R. J. Schmidt, *Ph.D.**

1. *Introduction.*

THE instability of a horizontal layer of fluid, when heated from below, has been considered by a number of writers †. Definite criteria for the onset of instability have been found mathematically ‡ and theoretical prediction has been verified by experiment §.

The problem of the stability of a layer of fluid bounded by two concentric spherical surfaces is considered in this paper. It is believed by some investigators that the motion ensuing from the instability of such a layer occurs in the sun ‖, and that such motion is symmetrical about the axis of the sun. It will be assumed in this paper that any motion that occurs is symmetrical about an axis.

The work will only be carried out for the case when the fluid is incompressible. It is, however, natural to assume, and it has been shown mathematically by Jeffreys ¶ that, if the difference between the actual and the adiabatic temperature gradients is substituted for the temperature gradient given by the results for a non-compressible fluid, the investigations hold for a compressible fluid to a first approximation.

2. *The Equation governing the Motion of the Fluid.*

Take spherical polar coordinates, the origin being the centre of the spherical surfaces. Let r be the length of the radius vector and θ the angle the radius vector makes with some fixed line passing through the centre of the sphere. Let u and v be the component velocities of the fluid along and perpendicular to the radius vector. Let T be the temperature at any point of the fluid, and let ρ, ν, κ, c, and α be the density, kinematic viscosity,

* Communicated by the Author.

† Benard, *Ann. Chim. Phys.* vol. xxiii. p. 62 (1901).

‡ Rayleigh, Phil. Mag. vol. xxxii. p. 529 (1916). Jeffreys, Phil. Mag. vol. ii. p. 833 (1926); Proc. Roy. Soc. A, vol. cxviii. p. 195 (1928). Low, Proc. Roy. Soc. A, vol. cxxv. p. 180 (1929).

§ Schmidt and Milverton, Proc. Roy. Soc. A, vol. cliii. p. 586 (1935).

‖ Bierknes, 'Astrophysical Journal,' vol. lxiv. (1926).

¶ Proc. Camb. Phil. Soc. vol. xxvi. p. 170 (1930).

thermal conductivity, specific heat, and expansion coefficient of the fluid. Let suffix zero attached to any of them indicate its value in the undisturbed state and an accent the departure from the undisturbed value; so that, for example, $\rho=\rho_0+\rho'$.

If we assume axial symmetry, and that any motion that occurs is in planes through the axis, the equations governing the state of the fluid are :

(*a*) the two equations of motion

$$\frac{Du}{Dt}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{\partial V}{\partial r}+\nu\left(\nabla^2 u-\frac{2u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial\theta}-\frac{2v\cot\theta}{r^2}\right)*, \quad . \quad (2.1)$$

$$\frac{Dv}{Dt}+\frac{uv}{r}=-\frac{1}{r\rho}\frac{\partial p}{\partial\theta}-\frac{1}{r}\frac{\partial V}{\partial\theta}+\nu\left(\nabla^2 v+\frac{2}{r^2}\frac{\partial u}{\partial\theta}-\frac{v}{r^2\sin^2\theta}\right); \quad . \quad . \quad . \quad (2.2)$$

(*b*) the equation of continuity

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)=0; \quad (2.3)$$

(*c*) Fourier's heat conduction and convection equation

$$\frac{DT}{Dt}=\frac{1}{c}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\kappa\frac{\partial T}{\partial\theta}\right)\right\}; \quad (2.4)$$

and (*d*) the equation connecting the density and temperature of the fluid, which we will take to be

$$\rho=\rho_c(1-\alpha T), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.5)$$

where ρ_c is the density at some given temperature, say, the mean temperature of the fluid.

For a liquid at rest, surrounding a heated sphere, and acted upon by gravity only, these equations reduce to

$$\frac{1}{\rho_0}\frac{dp_0}{dr}+g\rho_0=0 \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

and

$$\frac{d}{dr}\left(r^2\kappa\frac{dT_0}{dr}\right)=0. \quad . \quad . \quad . \quad . \quad . \quad (2.7)$$

* Ramsay, ' Hydrodynamics,' fourth edition, p. 374.

If the thermal conductivity is constant throughout, the fluid equation (2.7) gives us

$$T_0 = \frac{\beta}{r} + B. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.8)$$

We wish to find at what value of β the steady state of the fluid becomes unstable.

3. *The Equations of Disturbed Motion.*

Suppose that the departure from the steady state is small, and consider marginal stability so that $\frac{\partial}{\partial t} = 0$. Neglecting squares of small quantities the equations (2.1), (2.2), (2.3), (2.4), (2.5), become

$$0 = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial r}(p_0 + p') - g + \nu\left\{\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) - \frac{2u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta} - \frac{2v\cot\theta}{r^2}\right\} \quad . \quad . \quad . \quad (3.1)$$

$$0 = -\frac{1}{r\rho_0}\frac{\partial p'}{\partial\theta} + \nu\left\{\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{2}{r^2}\frac{\partial u}{\partial\theta} - \frac{v}{r^2 \sin^2\theta}\right\} \quad . \quad . \quad . \quad (3.2)$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(\rho_0 u r^2) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\rho_0 v \sin\theta), \quad . \quad . \quad . \quad (3.3)$$

$$u\frac{\partial_0 T}{\partial T} = \frac{\kappa}{c}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{dT'}{dr}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T'}{\partial\theta}\right)\right]. \quad . \quad . \quad (3.4)$$

Since ρ'/ρ_0 is small, the first two terms in equation (3.1) may be written

$$\left(\frac{1}{\rho_0} - \frac{\rho'}{\rho_0^2}\right)\left(\frac{\partial p_0}{\partial r} + \frac{\partial p'}{\partial r}\right) + g = -\frac{1}{\rho_0}\frac{\partial p'}{\partial r} - \frac{g\rho'}{\rho_0}, \quad (3.5)$$

using (2.6) and neglecting second order quantities.

4. *Reduction of the Equations.*

Let $P_n(\cos\theta)$ be any solution of Legendre's equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right)+n(n+1)\,y=0, \quad . \quad . \quad (4.1)$$

and assume that

$$\left.\begin{aligned}(u,\ p',\ \rho',\ T')&=(u_1,\ p_1,\ \rho_1,\ T_1)\,P_n(\cos\theta),\\ v&=v_1\sin\theta P_n'(\cos\theta),\end{aligned}\right\}\ .\ .\ .\ .\ (4.2)$$

where u_1, v_1, p_1, ρ_1, T_1 are functions of r only. We restrict n to integral values, as otherwise $P_n(\cos\theta)$ is infinite when $\theta=\pm\pi/2$.

On making use of (4.1) and of the relations

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}\{\sin\theta\,P_n'(\cos\theta)\}\right]-\frac{P_n'(\cos\theta)}{\sin\theta}$$
$$=-n(n+1)\sin\theta\,P_n'(\cos\theta), \quad . \quad (4.3)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\{\sin^2\theta\,P_n'(\cos\theta)\}=n(n+1)\,P_n(\cos\theta), \quad (4.4)$$

and replacing ρ_0 by ρ_c, the equations (3.1), (3.2), (3.3), (3.4), and (2.5) become

$$0=-\frac{1}{\rho_c}\frac{dp_1}{dr}-\frac{g\rho_1}{\rho_0}+\nu\left[\frac{d^2u_1}{dr^2}\right.$$
$$\left.+\frac{2}{r}\frac{du_1}{dr}-\frac{n(n+1)u_1}{r^2}-\frac{2u_1}{r^2}-\frac{2n(n+1)v_1}{r^2}\right],$$
$$. \quad . \quad . \quad (4.5)$$

$$0=\frac{p_1}{r\rho_0}+\nu\left[\frac{d^2v_1}{dr^2}+\frac{2}{r}\frac{dv_1}{dr}-\frac{n(n+1)v_1}{r}-\frac{2u_1}{r^2}\right], \quad . \quad (4.6)$$

$$0=\frac{1}{r}\frac{d}{dr}(r^2u_1)+n(n+1)v_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.7)$$

$$-u_1\beta=\frac{\kappa}{c}\left[\frac{d}{dr}\left(r^2\frac{dT_1}{dr}\right)-n(n+1)T_1\right], \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

$$\rho_1=-\rho\alpha T_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.9)$$

Since n is integral, n^2+n is necessarily positive, and we shall denote it by p^2.

Equations (4.7) and (4.8) give us

$$v_1=-\frac{\kappa}{p^2c\beta}\left\{r^3\frac{d^3T_1}{dr^3}+6r^2\frac{d^2T_1}{dr^2}-(p^2-6)r\frac{dT_1}{dr}-2p^2T_1\right\}.$$
$$. \quad . \quad . \quad (4.10)$$

Eliminating p_1, ρ_1, T_1 between the equations (4.5), (4.6), (4.9), and (4.10), we get

$$\frac{g\alpha p^2}{\nu}.\,T_1 = r^2\frac{d^4u_1}{dr^4} + 8r\frac{d^3u_1}{dr^3} + (12-2p^2)\frac{d^2u_1}{dr^2} - \frac{4p^2}{r}\frac{du_1}{dr} + \frac{p^2(p^2-2)}{r^2}u_1. \quad . \quad . \quad (4.11)$$

Let R be the mean radius of the fluid layer, and let $r=\xi R$,

$$\frac{g\alpha c\beta R^2}{\kappa\nu} = \lambda_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.12)$$

Eliminating u_1 between the equations (4.7) and (4.11), and using the notation of (4.12), we finally get

$$\xi^6\frac{d^6T_1}{d\xi^6} + 18\xi^5\frac{d^5T_1}{d\xi^5} + (96-3p^2)\xi^4\frac{d^4T_1}{d\xi^4} + (168-24p^2)\xi^3\frac{d^3T_1}{d\xi^3} + (72-36p^2+3p^4)\,\xi^2\frac{d^2T_1}{d\xi^3} + (-12p^2+6p^4)\xi\,\frac{dT_1}{d\xi} + (\lambda_1 p^2+2p^4-p^6)T_1 = 0. \quad . \quad . \quad . \quad (4.13)$$

5. *Boundary Conditions.*

As boundaries we shall take perfectly conducting surfaces, exerting no tangential force, the velocity normal to the boundary being zero. At such a boundary we have

$$T'=0,\ u=0,\ \mu\left\{\frac{1}{r}\frac{\partial u}{\partial\theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right\} = 0\ *. \quad . \quad (5.1)$$

The expression on the left-hand side of the last equation is the tangential stress.

Making use of the equations (4.2), (4.7), (4.8), these equations reduce to

$$T_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.2)$$

$$\xi\frac{d^2T_1}{d\xi^2} + 2\frac{dT_1}{d\xi} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.3)$$

$$\xi^2\frac{d^4T_1}{d\xi^4} + 8\xi\frac{d^4T_1}{d\xi^3} + 12\frac{d^2T_1}{d\xi^3} = 0 \quad . \quad . \quad . \quad (5.4)$$

We know from the case of a horizontal layer of fluid † that the nature of the results will not be affected by making

* Ramsay (*loc. cit.*). † Jeffreys and Low (*loc. cit.*).

the boundaries rigid. The actual numerical result for λ will, however, be about three times as great for rigid boundaries as for free surfaces.

6. *Method of Solution.*

In order that there may be a solution to (4.13) satisfying these boundary conditions there must be some relation between λ_1 and p. For some value of p, λ_1 will be a minimum, and it is this minimum value of λ_1 that we wish to find in order to know when the fluid becomes unstable.

The point $\xi=1$ is an ordinary point for the differential equation, and hence the solution can be expanded in a power series in the neighbourhood of $\xi=1$. The equation was solved by finding a series of polynomials that satisfied the boundary conditions and combining them together so that they nearly as possible satisfied the equation.

We shall suppose that the boundaries of the fluid are the spherical surfaces $\xi=1+d$, $\xi=1-d$, so that the depth h of the fluid layer is $2Rd$.

7. *First Approximation.*

Let $\eta=1+\xi$, so that the boundaries become $\eta=\pm d$. If we suppose T_1 to be of the form

$$A_0+A_1\eta+A_2\eta^2+A_3\eta^3+A_4\eta^4+A_5\eta^5+A_6\eta^6, \quad . \quad (7.1)$$

it can be verified that T_1 will satisfy the boundary conditions (5.2), (5.3), and (5.4) if

$$\left.\begin{aligned}
A_0&=183d^6-2{,}113d^8+5{,}629d^{10}-4{,}851d^{12},\\
A_1&=-208d^6+1{,}536d^8-2{,}352d^{10},\\
A_2&=-225d^4+2{,}839d^6-8{,}523d^8+8{,}085d^{10},\\
A_3&=+256d^4-1{,}856d^6+3{,}136d^8,\\
A^4&=+45d^2-827d^4+3{,}343d^6-3{,}969d^8,\\
A_5&=-48d^2+320d^4-784d^6,\\
A^6&=-3+101d^2-449d^4+735d^6.
\end{aligned}\right\} \quad . \quad (7.2)$$

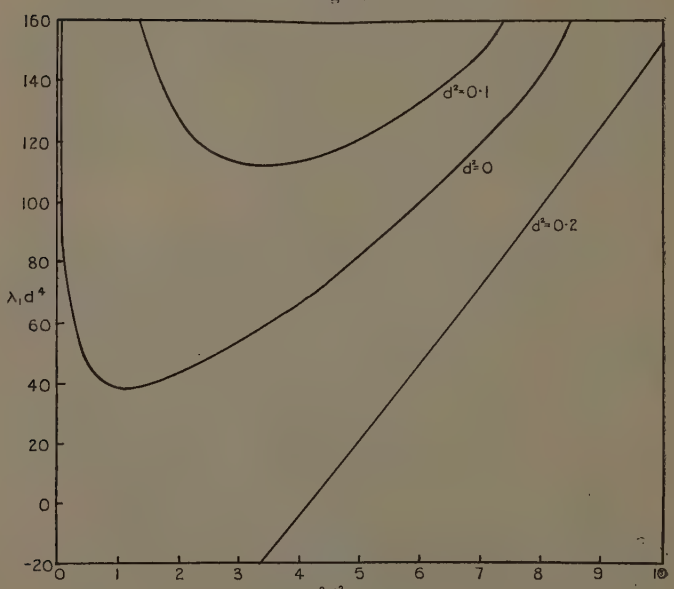
Substituting the value of T_1 so obtained in the differential equation it is found that the constant term will vanish if

$$\begin{aligned}
720A_5+18.120A_5+(96-3p^2)24A_4+(168-24p^2)6A_3\\
+(3p^4-36p^2+72)2A_2+(6p^4-12p^2)A_1\\
+(\lambda_1p^2+2p^2-p^4)A_0=0 \quad . \quad . \quad . \quad . \quad (7.3)
\end{aligned}$$

Substituting the values of the A's, and rearranging the terms, we get

$$\left.\begin{aligned} &2{,}160-72{,}720d^2+1{,}311{,}840d^4-5{,}076{,}000d^6+7{,}210{,}800d^8 \\ &\qquad -1{,}164{,}240d^{10} \\ &+p^2d^2[3{,}240-38{,}880d^2+175{,}344d^4-429{,}408d^6 \\ &\qquad +553{,}896d^{10}], \\ &+p^4d^4[1{,}350-16{,}152d^2+46{,}148d^4-45\,656d^6 \\ &\qquad +9{,}702d^8], \\ &+(p^6d^6-\lambda_1p^2d^6)[183-2{,}113d^2+5\,629d^{10} \\ &\qquad -4{,}851d^{12}.=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7.4)$$

Fig. 1.



Equation (7.4) is the desired relation between λ_1, p, and d, and a graph of $\lambda_1 d^4$ plotted against p^2d^2 for various values of d is given in fig. 1, the values required being found from the equation. It is obvious from its form that the graph for $d^2=0\cdot2$ cannot represent a true physical relation. This type of curve, with $\lambda_1 d^4$ tending to minus

infinity when p^2d^2 tends to zero first occurs for $d^2=0{\cdot}122$, when A_0 changes sign. It would appear that if $d>\sqrt{0{\cdot}122}$, the type of disturbance postulated cannot occur. This is a rather surprising result.

Let the value of

$$16\lambda_1 d^4 (=g\alpha c\beta R^4/\kappa\nu R^2),$$

at its minimum value for persistence of a disturbance, be denoted by λ, and let $\xi=2d/(1+d)$, so that ξ is the ratio of the depth of the fluid layer to the outer radius of the layer. A graph of λ and the corresponding value of n plotted against ξ is given in fig. 2. Since n must be integral the closest integer to the value given by the curve must always be taken.

8. *Further Approximation.*

Owing to the surprising nature of the results given by the first approximation for ξ large, a further approximation was carried out, but owing to the length of the computations required, only a few critical points have been found. Suppose now

$$T_1=\sum_{n=0}^{6} A_n\eta^n+y\sum_{n+1}^{7} B_n\eta^n+\delta\sum_{n+2}^{8} c_n\eta^n \quad . \quad . \quad (8.1)$$

The first set of six terms chosen as in (7.2) satisfy the boundary conditions. If

$$\begin{aligned}
B_1 &= -9{,}455d^8+70{,}073d^{10}-146{,}789d^{12}+99{,}099d^{14},\\
B_2 &= +7{,}280d^8-30\,528d^{10}+32{,}208d^{12},\\
B_3 &= +14{,}945d^6-119{,}335d^8+265{,}547d^{10}-189{,}189d^{12},\\
B_4 &= -10{,}240d^6+42{,}560d^8-46{,}656d^{10},\\
B_5 &= -6{,}405d^4+59{,}827d^6-146{,}903d^8+114{,}345d^{10},\\
B_6 &= +2{,}960d^4-12{,}032d^6+14{,}448d^8,\\
B_7 &= +915d^2-10{,}565d^4+28\,145d^6-24{,}255d^8,\\
C_2 &= +9{,}703d^{10}-697{,}673d^{12}+1{,}452{,}749d^{14}-709{,}995d^{16},\\
C_3 &= +130{,}656d^{10}+273{,}280d^{12}-655{,}072d^{14},\\
C_4 &= -17{,}825d^8+1{,}033{,}695d^{10}-2{,}493{,}547d^{12}\\
&\qquad +1{,}474{,}605d^{14},\\
C_5 &= -214{,}432d^8-378{,}816d^{10}+1{,}081{,}696d^{12},\\
C_6 &= +10{,}013d^6-306{,}995d^8+1{,}118{,}943d^{10}-993{,}993d^{12},\\
C_7 &= +83{,}776d^6+105{,}536d^8-426{,}624d^{10}.\\
C_8 &= -1{,}891d^4-29{,}027d^6-78{,}145d^8+229{,}383d^{10},
\end{aligned}$$

. . . (8.2)

the second and third set of six terms separately satisfy the boundary conditions.

With these values for the coefficients the value of T_1 given by (8.1) satisfies the boundary conditions, whatever the value of γ and δ. γ and δ were chosen, so that when the above value of T_1 was substituted into the equation (4.13), the constant term and the coefficients of η and η^2 vanish identically. If

$$A_n+\gamma B_n+\delta c_n=a_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.3)$$

TABLE.

$d^2=0$.

p^2d^2	1	1·5	2
$\lambda_1 d^4$ (first approximation) . . .	37·9	38·9	42·4
$\lambda_1 d^4$ (second approximation) .	42·02	41·88	44·76

$d^2=0{\cdot}01$.

p^2d^2	1
$\lambda_1 d^4$ (first approximation) . . .	35·6
$\lambda_1 d^4$ (second approximation) .	35·07

$d^2=0{\cdot}1$.

p^2d^2	1	1·6	2	3
$\lambda_1 d^4$ (first approximation) . . .	196·4	143·0	127·4	112·6
$\lambda_1 d^4$ (second approximation) .	103	63·15	58·6	56

the substitution of $\sum_{n=0}^{8} an\,\eta^n$ for T_1 in the equation (4.13) gives us

$$\left.\begin{aligned} &720a_6+2{,}160a_5+2{,}304a_4+1{,}008a_3+144a_2,\\ &\quad -p^2(72a_4+144a_3+72a_2+12a_1),\\ &\quad +p^4(6a_2+6a_1+2a_0)+(\lambda_1 p^2-p^6)a_0=0, \end{aligned}\right\} \quad . \quad . \quad (8.4)$$

from the vanishing of the constant term.

Since the coefficient of η vanishes, we get

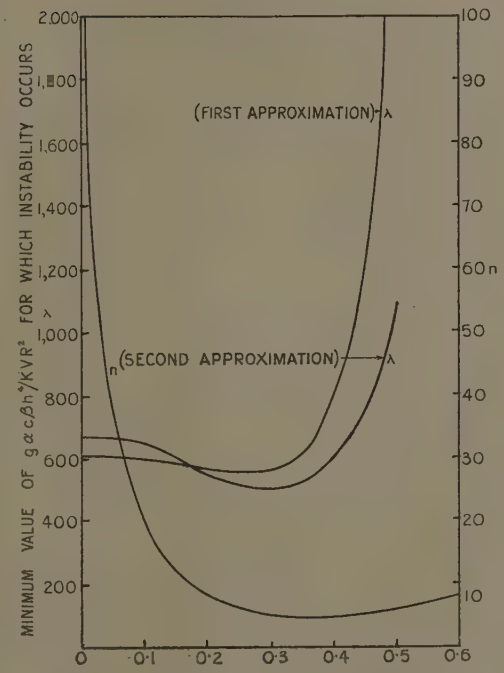
$$\left.\begin{aligned} &5{,}040a_7+17{,}280a_6+22{,}320a_5+13{,}248a_4+3{,}456a_3+288a_2,\\ &\quad -p^2(360a_5+864a_4+648a_3+168a_2+12a_1,\\ &\quad +p^4(18a_3+24a_2+8a_1)+(\lambda_1 p^2-p^6)a_1=0, \end{aligned}\right\}$$

$$. \quad . \quad . \quad (8.5)$$

and since the coefficient of η^2 vanishes, we get

$$20{,}160a_8+75{,}600a_7+110{,}160a_6+77{,}760a_5+26{,}784a_4$$
$$+3{,}888a_3+144a_2-p^2(1{,}080a_6+2{,}880a_5+2{,}592a_4$$
$$+900a_3+96a_2+p^4(36a_4+54a_3+20a_2)$$
$$+(\lambda_1 p^2-p^6)a_2=0 \quad . \quad . \quad . \quad (8.6)$$

Fig. 2.



The equations (8.3), (8.4), and (8.5) were evaluated when $d^2=0$, 0·01, and 0·1. From the resulting equations λ_1 was determined for various values of p^2d^2, by eliminating γ and δ from the three equations so obtained and solving the resulting cubic for λ_1. The results are tabulated in the table which also gives the results of the first approximation for comparison. Estimated values of λ given by the second approximation are given in fig. 2.

The curve for n plotted against ξ is scarcely altered by the further approximation.

In all the results obtained, both for the first and second approximations, the straight lines $p^2d^2=0$, and $\lambda_1 d^4=p^2d^2$ are each asymptotes to the curves of $\lambda_1 d^4$ plotted against p^2d^2.

9. *The Motion.*

Now v, the component of velocity perpendicular to the radius vector is equal to $v_1 \sin\theta \,.\, P_n'(\cos\theta)$, and hence vanishes when $\theta=0$, π, or when $P_n'(\cos\theta)=0$. Now $P_n'(\cos\theta)$ has $n-1$ roots lying between 0 and π, and hence there is a form of cellular motion, there being n cells.

It has been found experimentally, both by Benard * and by the author in collaboration with O. A. Saunders † that in the case of a horizontal layer of fluid heated from below, that some time after instability has occurred, the wave-length of the vortices formed is greater than that given by the theory for the first vortex that will occur. It is, however, not surprising that the wave-length that occurs in the motion for any λ_1 is the wave-length given by the theory for that particular value of λ_1 and not the wave-length for the minimum value of λ_1 as was expected by some investigators. The theory gives two values for the wave-length for any particular value of λ_1 above the minimum, but experiment would appear to show that it is the greater of these that actually occurs. We might, therefore, expect in practice to see smaller values of n occur than those given by fig. 2.

It is a matter of some interest to see, in some particular case, what the depth of the fluid would be if $n=2$, so that the motion consists of fluid ascending or descending at the pole of some globe and descending or ascending at its equator. In the Unsöld layer of the sun, the actual temperature gradient is greater than the adiabatic, and we can therefore apply the previous results. If $n=2$, $p^2=6$. If the layer is thin, d^2 is small and equation (7.4) becomes

$$\frac{g\alpha c\beta h^6}{\kappa\nu \mathrm{R}^2}=126. \quad \ldots\ldots \quad (9.1)$$

* *Loc. cit.*
† Not yet published.

Taking Plaskett's * values, $g=2{\cdot}7\times10^4$ cm. sec.$^{-2}$, $\kappa/c\beta=\mathrm{V}=2$ cm.^{2} sec.$^{-1}$, $\alpha=3{\cdot}7\times10^{-3}$ per degree, $\beta=2{\cdot}7\times10^{-5}$ degrees cm.$^{-1}$, and taking $\mathrm{R}=7{\cdot}0\times10^{10}$ cm., this relation gives h to be about a third of a kilometre. This result is far too small to fit in with the vortex theory of sunspots.

The true equation for h gives an infinite number of values for h, and in the above work we have taken the minimum root. It is conceivable, however, that one of the larger roots should be taken.

10. *Results for a Horizontal Layer.*

For a horizontal layer, we have $d=0$, and the present theory then gives $\lambda=666$, $p^2d^2=1{\cdot}28$. Now β/R^2 is the temperature gradient at the mid-point of the fluid, and hence we have the product of $g\alpha ch^4/\kappa\nu$ and the temperature gradient$=666$. This compares favourably with the exact value of 651 calculated by Rayleigh.

Now, when n is large,

$$\mathrm{P}_n(\cos\theta)=\left(\frac{2}{n\pi\sin\theta}\right)^{\frac{1}{2}}\sin\left(n+\tfrac{1}{2}\theta+\frac{\pi}{4}\right)+0\left(\frac{1}{n^{3/2}}\right), \quad (10.1)$$

θ being confined to the interval $(\epsilon, \pi-\epsilon)$.

If x is the distance from the axis measured along the middle layer, we have $x=\mathrm{R}\theta$, $\sin\left(n+\frac{1}{2}\theta+\frac{\pi}{4}\right)$ becomes $\sin nx/\mathrm{R}=\sin lx$ if $l=\frac{n}{\mathrm{R}}$; now $p^2d^2=(n^2+n)d^2=1{\cdot}28$;

n being large, n is negligible compared with n^2, and we have $nd=\sqrt{1{\cdot}28}$. *i. e.*,

$$l\mathrm{R}\ \mathrm{R}h/2\mathrm{R}=1{\cdot}14, \quad lh=2{\cdot}28.$$

This is to be compared with the exact value of 2·22, calculated by Rayleigh.

The author is indebted to Prof. S. Chapman for suggesting the investigation.

* M. N. R. A. S. 1936.

XXII. *Note on the Maximum Correlation Coefficient between Two Series whose Values have been determined at Equal Intervals. By* J. T. Morrison, *Emeritus Professor of Mathematical Physics, University of Stellenbosch, South Africa* *.

1. THE calculation of the correlation coefficient between two series observed at equal intervals of a variable, and especially between two time-series, is a common and necessary statistical operation in many branches of science, particularly in meteorology and economics. For instance, it may be desirable to correlate the daily barometric changes at one place in a country with those occurring on the same day or a succeeding or preceding day at another place ; or to correlate the monthly changes of wholesale prices of commodities with the changes in unemployment. In such cases the suspicion must frequently arise that the maximum correlation really occurs between the values of the first series and values of the second advanced or retarded relatively to the first by a fraction of an interval ; and, as the question of a causal connection between the two series is bound up intimately with this question of a possible advance or lag, the determination of the existence and amount of advance or lag, and of the corresponding maximum correlation coefficient, is a matter of considerable importance. In most cases it may be impossible to supply or procure intermediate values, but the following simple method, which uses the original values, seems to lead to the required determinations.

2. Let corresponding values of the departures of the two series from their means be

$$a_{-1}, \quad a_0, \quad a_1, \quad a_2, \ \ldots \ a_n, \quad a_{n+1}, \text{ etc.},$$

and

$$b_{-1}, \quad b_0, \quad b_1, \quad b_2, \ \ldots \ b_n, \quad b_{n+1}, \text{ etc.},$$

and let us correlate n successive a values with n successive b values. An inspection or rough calculation of

$$\overset{n}{\underset{1}{\mathrm{S}}}(a_p b_{p-2}), \quad \overset{n}{\underset{1}{\mathrm{S}}}(a_p b_{p-1}), \quad \overset{n}{\underset{1}{\mathrm{S}}}(a_p b_p), \quad \overset{n}{\underset{1}{\mathrm{S}}}(a_p b_{p+1}), \text{ etc.},$$

* Communicated by the Author.

will show the interval in which the maximum correlation coefficient will lie. Suppose, for simplicity, that

$$\overset{n}{\underset{1}{\mathrm{S}}}(a_p b_p) \quad \text{and} \quad \overset{n}{\underset{1}{\mathrm{S}}}(a_p b_{p+1})$$

have the greatest values of the product-sums.

The proposed method is to write down the correlation coefficient r_x of a_p with $(xb_p+\overline{1-x}\,b_{p+1})$ from $p=1$ to $p=n$, where x is a positive fraction less than unity, and to find what value of x makes $\dfrac{dr_x}{dx}$ zero.

As a second determination we can associate $(x'a_p+\overline{1-x'}a_{p+1})$ with b_{p+1} from $p=1$ to $p=n$ to determine $r_{x'}$ and find the value of x' which will make $\dfrac{dr_{x'}}{dx'}$ zero.

Let $\overset{n}{\underset{p=1}{\mathrm{S}}}(a_p b_p)$ be written simply $\mathrm{S}(a_p b_p)$, and so for the other sums of products.

Then

$$r_x=\frac{x\mathrm{S}(a_p b_p)+(1-x)\mathrm{S}(a_p b_{p+1})}{\sqrt{\mathrm{S}(a_p^2)\mathrm{S}\{xb_p+(1-x)b_{p+1}\}^2}}$$

$$=\frac{x\mathrm{S}(a_p b_p)+(1-x)\mathrm{S}(a_p b_{p+1})}{\sqrt{\mathrm{S}(a_p^2)[x^2\{\mathrm{S}(b_p^2)+\mathrm{S}(b_{p+1}^2)-2\mathrm{S}(b_p b_{p+1})\}+2x\{\mathrm{S}(b_p b_{p+1})-\mathrm{S}(b_{p+1}^2)\}+\mathrm{S}(b_{p+1}^2)]}} \quad \ldots \quad (1)$$

Hence $\dfrac{dr_x}{dx}=0$ if

$$x=\frac{\mathrm{S}(a_p b_p)\mathrm{S}(b_{p+1}^2)-\mathrm{S}(a_p b_{p+1})\mathrm{S}(b_p b_{p+1})}{\mathrm{S}(a_p b_p)\{\mathrm{S}(b_{p+1}^2)-\mathrm{S}(b_p b_{p+1})\}-\mathrm{S}(a_p b_{p+1})\{\mathrm{S}(b_p b_{p+1})-\mathrm{S}(b_p^2)\}} \quad \ldots \quad (2)$$

This is the required value of x to give a maximum correlation coefficient on the approximate assumption that the value of b belonging to the fraction x of the pth interval and the fraction $(1-x)$ of the $(p+1)$th interval is $xb_p+(1-x)b_{p+1}$. This value of x may be called $x_{\max}$.

The corresponding value of r_x is written down from (1), namely,

$$r_{x_{\max}} = \frac{x_{\max}\{S(a_pb_p)-S(a_pb_{p+1})\}+S(a_pb_{p+1})}{\sqrt{S(a_p^2)[x^2_{\max}\{Sb_p^2)+S(b^2_{p+1})-2S(b_pb_{p+1})\} \quad -2x_{\max}\{S(b^2_{p+1})-S(b_pb_{p+1})\}+S(b^2_{p+1})]} \quad . \quad . \quad . \quad (3)$$

Similarly, if we associate $\{x'a_p+(1-x')a_{p+1}\}$ with b_{p+1} to determine $r_{x'}$, we find

$$r_{x'} = \frac{x'S(a_pb_{p+1})+(1-x')S(a_{p+1}b_{p+1})}{\sqrt{S(b^2_{p+1})[x'^2\{S(a_p^2)+S(a^2_{p+1})-2S(a_pa_{p+1})\}-2x'\{S(a^2_{p+1}) \quad -S(a_pa_{p+1})\}+S(a^2_{p+1})]} \quad . \quad . \quad . \quad (4)$$

Hence r_x' reaches a maximum when

$$x' = \frac{S(a_{p+1}b_{p+1})S(a_pa_{p+1})-S(a_pb_{p+1})S(a^2_{p+1})}{S(a_{p+1}b_{p+1})\{S(a_pa_{p+1})-S(a_p^2)\}-S(a_pb_{p+1})\{S(a^2_{p+1}) \quad -S(a_pa_{p+1})\}} = x'_{\max}. \quad . \quad . \quad . \quad (5)$$

The corresponding value of $r_{x'}$ is

$$\frac{x'_{\max}\{S(a_pb_{p+1})-S(a_{p+1}b_{p+1})\}+S(a_{p+1}b_{p+1})}{\sqrt{S(b^2_{p+1})[x'^2_{\max}\{S(a_p^2)+S(a^2_{p+1})-2S(a_pa_{p+1})\} \quad -2x'_{\max}\{S(a^2_{p+1})-S(a_pa_{p+1})\}+S(a^2_{p+1})]} \quad . \quad . \quad . \quad (6)$$

Since the assumption made here, namely, that the fraction x' of the pth interval and $(1-x')$ of the $(p+1)$th interval has a value of a equal to $x'a_p+(1-x')a_{p+1}$, is again only a first approximation, it cannot be expected that $x_{\max}$ will be accurately equal to $(1-x'_{\max})$, as it would be if this and the previous approximate assumption were accurately true. We may take the mean of $\frac{x'_{\max}+1-x_{\max}}{2}$ as the mean lag of the b observations on the a observations, and $\frac{r_{x\max}+r_{x'\max}}{2}$ as the mean maximum coefficient of correlation.

3. If the correlated numbers $a_1, a_2, \ldots a_{n+1}, b_1, b_2 \ldots b_{n+1}$ have not been reduced to departures from the means, the formulæ given above must be modified in the usual way. Thus for $S(a_p^2)$ we must substitute $S(a_p^2) - na_p^2$ wherever the former occurs. Similarly $S(a_p b_p)$ must be replaced by $S(a_p b_p) - n\bar{a}_p\bar{b}_p$, and so for the other sums of squares and sums of products.

4. The following example will illustrate the proposed method and also the degrees of approximation to equality

Date ..	Sept. 30	Oct. 1	2	3	5.	5	6
a	−153	192	−253	245	249	−473	12
b	−163	8	66	−158	571	−434	−62

Date ..	Oct. 7	8	9	10	11	12	13
a	−63	243	−18	−196	272	−247	−95
b	−71	226	18	−352	438	−359	60

Date ..	Oct. 14	15	16	17	18	19	20
a	−104	−117	479	−79	−94	145	209
b	−223	−80	434	68	−313	130	418

Date ..	Oct. 21	22	23	24	25	26	27
a	−262	−166	7	−222	423	18	158
b	−34	−289	−116	−293	450	22	64

Date ..	Oct. 28	29	30	31	Nov. 1
a	86	−346	38	−101	−188
b	284	−188	−291	82	−134

of x'_{max} and $(1-x_{max})$ and of $r_{1\,max}$ and $r_{1'\,max}$, which are likely to occur in practice. The two series to be correlated are the changes of barometric pressure which occurred at Port Elizabeth (long. E. 25° 37′, lat. S. 34° 9′), and Durban (long. E. 30° 51′, lat. S. 29° 48′), seaports in South Africa, on successive days from Sept. 30, 1932,

to Nov. 1, 1932. At all South African meteorological stations barometric readings are taken once daily at 8.30, S.A. Mean Time, and the change set opposite each date in the table is that which occurred between 8.30 on that date and 8.30 next day. The numbers $a_0a_1 \ldots . a_{31}a_{32}$ are the barometric changes in thousandths of an inch which occurred at Port Elizabeth on Sept. 30, Oct. 1 . . . . 31, Nov. 1; while the numbers $b_0,\ b_1 \ldots b_{31},\ b_{32}$ are the changes on the same dates at Port Elizabeth.

Here

$$\overset{31}{\mathrm{S}}(a_pb_{p-1})-n\bar{a}_p\bar{b}_{p-1}=-852100\ ;$$

$$\mathrm{S}(a_pb_p)-na_pb_p=1419600\ ;$$

$$\mathrm{S}(a_pb_{p+1})-na_pb_{p+1}=135200\ ;$$

$$\mathrm{S}(a_{p+1}b_{p+1}-n\bar{a}_{p+1}\bar{b}_{p+1}=1442600.$$

Hence it is clear that the maximum correlation will lie between $r_{a_pb_p}$ and $r_{a_pb_{p+1}}$.

$$\mathrm{S}(a_p^2)-n\bar{a}_p^2=1555200\ ;\quad \mathrm{S}(a_pa_{p+1})-na_pa_{p+1}=-455500\ ;$$

$$\mathrm{S}(a_{p+1}^2)-n\bar{a}_{p+1}^2=1549500.$$

$$\mathrm{S}(b_p^2)-nb_p^2=2189300\ ;\quad \mathrm{S}(b_pb_{p+1})-nb_pb_{p+1}=-749300\ ;$$

$$\mathrm{S}(b_{p+1}^2)-n\bar{b}_{p+1}^2=2207300.$$

Hence we find

$$x_{\max}=\cdot 704\ ;\ x'_{\max}=\cdot 273,$$

and thus the mean lag of the changes at Durban over those at Port Elizabeth is ·285 day or 6·83 hours. The corresponding correlation coefficients are

$$r_{x_{\max}}=\cdot 848\ ;\ r_{x'_{\max}}=\cdot 842\ ;$$

whence the mean maximum correlation coefficient may be taken to be ·845. For the correlation coefficient $r_{a_pb_p}$ we find the smaller value ·770.

Summary.

5. It is pointed out that, where two series of related observations have been determined at equal intervals, the maximum correlation will generally occur between values of the first series and values of the second advanced

or retarded, relatively to the first by a fraction of an interval. Formulæ are given for the fractional advance or lag that will give the maximum correlation, and for the maximum correlation coefficient itself. Application of the formulæ to the barometric changes observed daily during October 1932 at Port Elizabeth and Durban, South Africa (latter about 440 miles N.E. of former), shows a lag of the latter station of 6·83 hours and a maximum correlation coefficient of ·845 for that particular month.

XXIII. *Some Differential Equations involving Three-Term Recursion Formulas.* *By* E. Fisher *.

THERE is presented in this note a method of investigating linear differential equations, especially those of the second order which are more general than the hypergeometric and confluent hypergeometric. It is simply to solve as a difference equation the recursion formula satisfied by the coefficients of a power series solution. This method is tried out on the spheroidal, Lamé, and generalized spheroidal equations, where only three-term recursion formulas need be considered.

The spheroidal equation is

$$(z^2-1)\frac{d^2\omega}{dz^2}+2(a+1)z\frac{d\omega}{dz}+(c^2z^2-b)\omega=0.$$

Its even solution is

$$\mathrm{A}=\sum_{n=0}^{\infty}a_nz^{2n},$$

where

$$(n+1)(n+\tfrac{1}{2})a_{n+1}=\left[n(n+a+\tfrac{1}{2})-\frac{b}{4}\right]a_n+\frac{c^2}{4}a_{n-1}.$$

This difference equation has solutions

$$\lim_{n=\infty}n^{1-a}a_n'=1=\lim_{n=\infty}\left(-\frac{4}{c^2}\right)^n\Gamma(n+1)\Gamma(n+a+\tfrac{3}{2})a_n'',$$

where Γ is the Gamma function. In general, it is a_n'

* Communicated by E. L. Ince.

which satisfies $a_{-1}=0$. Similarly known are the spheroidal functions

$$\mathrm{B}=\sum_{n=0}^{\infty} b_n z^{2n+1}, \quad \mathrm{C}=\sum_{s=0}^{\infty} c_s(-v^2)^s, \quad \mathrm{D}=z\sum_{s=0}^{\infty} d_s(-v^2)^s,$$

$$\mathrm{E}=\sum_{s=0}^{\infty} c_{s-a}(-v^2)^{s-a}, \quad \mathrm{F}=z\sum_{s=0}^{\infty} d_{s-a}(-v^2)^{s-a},$$

where $$v^2=z^2-1.$$

If $\omega(a, b, c, z)$ denote a solution, another solution is

$$(z^2-1)^{-a}\omega(-a, b+2a, c, z);$$

therefore E, F can be derived from C, D.

For n, s integers we have

$$a_n=\frac{(-c^2/4)^n}{\Gamma(n+\frac{1}{2})}\sum_{s=0}^{n}\left(-\frac{4}{c^2}\right)^s\frac{\Gamma(s+1+a)}{\Gamma(n+1-s)}c_s,$$

$$b_n=\frac{(-c^2/4)^n}{\Gamma(n+\frac{3}{2})}\sum_{s=0}^{n}\left(-\frac{4}{c^2}\right)^s\frac{\Gamma(s+1+a)}{\Gamma(n+1-s)}d_s,$$

$$c_s=\frac{(c^2/4)^s}{\Gamma(s+1+a)}\sum_{n=0}^{s}\left(\frac{4}{c^2}\right)^n\frac{\Gamma(n+\frac{1}{2})}{\Gamma(s+1-n)}a_n,$$

$$d_s=\frac{(c^2/4)^s}{\Gamma(s+1+a)}\sum_{n=0}^{s}\left(\frac{4}{c^2}\right)^n\frac{\Gamma(n+\frac{3}{2})}{\Gamma(s+1-n)}b_n.$$

Substituting these in the above power series, we discover solutions

$$\mathrm{A}=\sum_{s=0}^{\infty} c_s\Gamma(s+1+a)\left(\frac{2z}{c}\right)^{s+1/2}\mathrm{J}^{(cz)}_{s-\frac{1}{2}},$$

$$\mathrm{B}=\sum_{s=0}^{\infty} d_s\Gamma(s+1+a)\left(\frac{2z}{c}\right)^{s+1/2}\mathrm{J}^{(cz)}_{s+\frac{1}{2}},$$

$$\mathrm{C}=\sum_{n=0}^{\infty} a_n\Gamma(n+\tfrac{1}{2})\left(-\frac{2v}{c}\right)^{n-a}\mathrm{J}^{(cv)}_{n+a},$$

$$\mathrm{D}=z\sum_{n=0}^{\infty} b_n\Gamma(n+\tfrac{3}{2})\left(-\frac{2v}{c}\right)^{n-a}\mathrm{J}^{(cv)}_{n+a},$$

where J is the Bessel function. All the above series converge, in general, only for $|z|$, or $|v|\leqslant|$.

But for characteristic values of b, $a''_{-1}=0$. Then $a_n=a''_n$ and

$$\mathrm{C}=\sum_{n=0}^{\infty} a_n\Gamma(n+\tfrac{1}{2})\left(-\frac{2v}{c}\right)^{n-a}\mathrm{J}^{(cv)}_{n+a}$$

is an integral function of v^2. For a second set of characteristic b, $b''_{-1}=0$. Then

$$\frac{\mathrm{D}}{\sqrt{1+v^2}} = \sum_{n=0}^{\infty} b_n \Gamma(n+\tfrac{3}{2}) \left(-\frac{2v}{c}\right)^{n-a} \mathrm{J}^{(cv)}_{n+a}$$

is an integral function of v^2. Therefore, except for a equal to a negative integer, the two sets of characteristic b have no value in common.

The expansion

$$\frac{(-c^2/4)^n}{\Gamma(n+1)} \sum_{s=0}^{\infty} \frac{\Gamma(s+1+a)c_s}{\Gamma(s+n+a+\frac{3}{2})}$$

satisfies the recursion formula of

$$a_n \text{ and } \xrightarrow[n\to\infty]{} \frac{c_0 \Gamma(a+1)}{\Gamma(n+1)\Gamma(n+a+\frac{3}{2})} \left(-\frac{c^2}{4}\right)^n [1+0(n^{-1})].$$

Therefore it represents a''_n, in general (because of convergence) only for $n \geqslant -1$. In the same way we find

$$c''_s = \frac{(c^2/4)^s}{\Gamma(s+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})a_n}{\Gamma(n+s+a+\frac{3}{2})},$$

$$b''_n = \frac{(-c^2 4/)^n}{\Gamma(n+1)} \sum_{s=0}^{\infty} \frac{\Gamma(s+1+a)\, d_s}{\Gamma(s+n+a+\frac{5}{2})},$$

$$d''_s = \frac{(c^2/4)^s}{\Gamma(s+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})b_n}{\Gamma(n+s+a+\frac{5}{2})}.$$

The b_n, d_s determine the odd characteristic b in the same way as the a_n, c_s do the even.

The latter are fixed, for example, as the roots of

$$\frac{c^2-b}{c^2(a+1)} = \frac{4}{c^2}\frac{c_1}{c_0} = \frac{4}{c^2}\frac{c''_1}{c''_0} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})a_n}{\Gamma(n+a+\frac{5}{2})} \Big/ \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})a_n}{\Gamma(n+a+\frac{3}{2})}.$$

By breaking off these sums at $n=5$, and neglecting in the resulting equation powers of b above the second, I have found $b=\cdot 320$ for $a=0$, $c=1$, and $b=\cdot 738$ for $a=1$, $c=2$: the correct values are $\cdot 319$, $\cdot 734$. Some such method of representing the lowest b might prove useful for a, c complex.

Assuming an even characteristic b, and using the expansion for a''_n, we find

$$\mathrm{A} = \sum_{s=0}^{\infty} c''_s \Gamma(s+a+1) \left(\frac{2}{cz}\right)^{s+a+1/2} \mathrm{J}^{(cz)}_{s+a+\frac{1}{2}}.$$

From the recurrence formula of $J_n(z)$ we see that this $\xrightarrow[z\to\infty]{}$

$$\Gamma(a+1)c_0''\left(\frac{2}{cz}\right)^{a+1/2}J^{(cz)}_{a+\frac{1}{2}}[1+0(z^{-1})].$$

And so, for characteristic b, and for $|\arg cz|<\pi$,

$$\frac{\sqrt{\pi}}{\Gamma(a+1)}\sum_{s=0}^{\infty}\frac{\Gamma(s+a+1)}{\Gamma(s+a+\frac{3}{2})}\frac{c_s''}{c_0''} \xleftarrow[z\to 0]{} A \xrightarrow[z\to\infty]{} \left(\frac{2}{cz}\right)^{a+1}\sin\left(cz-\frac{a\pi}{2}\right).$$

In the same way we find, for characteristic b,

$$B=\sum_{s=0}^{\infty}d_s''\Gamma(s+a+1)\left(\frac{2}{cz}\right)^{s+a+1/2}J^{(cz)}_{s+a+3/2},$$

$$C=\sum_{n=0}^{\infty}a_n''\Gamma(n+\tfrac{1}{2})\left(\frac{2}{cv}\right)^{n+a+1/2}J^{(cv)}_{n+a+1/2},$$

$$D=z\sum_{n=0}^{\infty}b_n''\Gamma(n+\tfrac{3}{2})\left(\frac{2}{cv}\right)^{n+a+3/2}J^{(cv)}_{n+a+3/2}.$$

The Lamé equation is

$$\frac{d^2L}{dx^2}+\frac{1}{2}\left(\frac{1}{x-e_1}+\frac{1}{x-e_2}+\frac{1}{x-e_3}\right)\frac{dL}{dx} = -\frac{a(a+1)x+c}{4(x-e_1)(x-e_2)(x-e_3)}L.$$

Denote $x_1=x-e_1$, $e_{12}=e_1-e_2$, $4c_1=a(a+1)e_1+c$, etc. A solution is $\sum_{n=0}^{\infty}A_nx_1^n$, where $A_{-1}=0$ and

$$e_{12}e_{13}(n+1)(n+\tfrac{1}{2})A_{n+1}+\left(n-1-\frac{a}{2}\right)\left(n-1+\frac{a+1}{2}\right)A_{n-1} = [(e_{21}+e_{31})n^2+c_1]A_n.$$

The solutions of this difference equation are

$$\lim_{n=\infty}e_{21}^n n^{3/2}A_n'=1, \qquad \lim_{n=\infty}e_{31}^n n^{3/2}A_n''=1.$$

With these asymptotic forms to guide us we discover

$$A_n'=\frac{\Gamma\left(n-\frac{a}{2}\right)\Gamma\left(n+\frac{a+1}{2}\right)}{\Gamma\left(-\frac{a}{2}\right)\Gamma\left(\frac{a+1}{2}\right)}\frac{e_{21}^{-n}}{\Gamma(n+1)}\sum_{s=0}^{\infty}\frac{\Gamma(s+\frac{1}{2})\alpha_s}{\Gamma(s+n+1)},$$

$$A_n'' = \frac{\Gamma\left(n-\frac{a}{2}\right)\Gamma\left(n+\frac{a+1}{2}\right)}{\Gamma\left(-\frac{a}{2}\right)\Gamma\left(\frac{a+1}{2}\right)} \frac{e_{31}^{-n}}{\Gamma(n+1)} \sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})a_s}{\Gamma(s+n+1)},$$

where α_s is derived from a_s by interchanging e_2 and e_3, and

$$e_{32}(s+1)(s+\tfrac{1}{2})a_{s+1} + e_{12}\left(s+\frac{a}{2}\right)\left(s-\frac{a+1}{2}\right)a_{s-1} = [e_{32}s^2 + e_{12}(s+\tfrac{1}{2})^2 + c_2]a_s,$$

whence

$$\lim_{s=\infty}\left(\frac{e_{23}}{e_{21}}\right)^s s^{1/2}a_s' = 1 = \lim_{s=\infty} s^{3/2}a_s''.$$

Suppose $|e_{23}| \geqslant |e_{31}|, |e_{12}|$.
Then

$$A_n = A_n' + \rho A_n'',$$

where ρ is fixed by $A_{-1}=0$; for example,

$$\frac{2c_1}{e_{12}e_{31}} = -\frac{A_1}{A_0} = -\frac{a^2+a}{4} \sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+2)}\left(\frac{\alpha_s}{e_{21}} + \rho\frac{a_s}{e_{31}}\right) \Big/ \sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}(\alpha_s + \rho a_s).$$

In actually calculating ρ we should use (say) A_6/A_5, because then the series involved converge much more rapidly.

Now, for x_1 small enough, the double series $\sum_{n=0}^{\infty} A_n x_1^n$ (or, for $|e_{23}| = |e_{31}|, |e_{12}|$, the series $\sum_{n=1}^{\infty} A_n x_1^n$) is absolutely convergent. And so

$$\sum_{n=0}^{\infty} A_n x_1^n = \sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \alpha_s \mathrm{F}\left(-\frac{a}{2}, \frac{a+1}{2}; s+1; \frac{x_1}{e_{21}}\right) + \rho \sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} a_s \mathrm{F}\left(-\frac{a}{2}, \frac{a+1}{2}; s+1; \frac{x_1}{e_{31}}\right),$$

where F is the hypergeometric series.

Likewise we find

$$A'_n = \frac{\Gamma\left(n-\frac{a}{2}\right)\Gamma\left(n+\frac{a+1}{2}\right)}{\Gamma\left(\frac{1-a}{2}\right)\Gamma\left(1+\frac{a}{2}\right)} \frac{e_{21}^{-n}}{\Gamma(n+\frac{1}{2})} \sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})\beta_s}{\Gamma(s+n+\frac{3}{2})},$$

$$A''_n = \frac{\Gamma\left(n-\frac{a}{2}\right)\Gamma\left(n+\frac{a+1}{2}\right)}{\Gamma\left(\frac{1-a}{2}\right)\Gamma\left(1+\frac{a}{2}\right)} \frac{e_{31}^{-n}}{\Gamma(n+\frac{1}{2})} \sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})b_s}{\Gamma(s+n+\frac{3}{2})},$$

where β_s is derived from b_s by interchanging e_2 and e_3, and

$$e_{32}(s+1)(s+\tfrac{1}{2})b_{s+1} + e_{12}\left(s+\frac{a+1}{2}\right)\left(s-\frac{a}{2}\right)b_{s-1} = [(e_{12}+e_{32})(s+\tfrac{1}{2})^2 + c_2]b_s.$$

Then

$$A_n = A'_n + \sigma A''_n,$$

where σ is fixed by $A_{-\frac{1}{2}}=0$, and a second Lamé function is

$$\sum_{n=0}^{\infty} A_{n+1/2} x_1^{n+1/2} = \left(\frac{x_1}{e_{21}}\right)^{1/2} \sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+2)} \beta_s F\left(\frac{1-a}{2},\ 1+\frac{a}{2};\ s+2;\ \frac{x_1}{e_{21}}\right) + \sigma\left(\frac{x_1}{e_{31}}\right)^{1/2} \sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+2)} b_s F\left(\frac{1-a}{2},\ 1+\frac{a}{2};\ s+2;\ \frac{x_1}{e_{31}}\right).$$

These series give the continuation of the Lamé functions for $x<\infty$. Suppose, for example, that $|e_{23}| > |e_{31}| > |e_{12}|$ and we wish to join $\sum_{n=0}^{\infty} A_n x_1^n$ to the solutions about $x=e_3$. Under known restrictions we have

$$F\left(-\frac{a}{2},\ \frac{a+1}{2};\ s+1;\ \frac{x_1}{e_{31}}\right) = \frac{\Gamma(s+1)\Gamma(s+\frac{1}{2})\left(\frac{x_1}{e_{31}}\right)^{s/2}}{\Gamma\left(s+1+\frac{a}{2}\right)\Gamma\left(s+\frac{1-a}{2}\right)} F\left(-\frac{a}{2},\ -s-\frac{a}{2};\ \tfrac{1}{2}-s;\ \frac{x_3}{x_1}\right)$$

$$+\frac{\Gamma(s+1)(\Gamma-s-\frac{1}{2})}{\Gamma\left(-\frac{a}{2}\right)\Gamma\left(\frac{a+1}{2}\right)}\left(\frac{x_3}{e_{13}}\right)^{1/2}\left(-\frac{x_3}{x_1}\right)^s$$

$$\times \mathrm{F}\left(1+\frac{a}{2},\ \frac{1-a}{2};\ s+\tfrac{3}{2};\ \frac{x_3}{e_{13}}\right).$$

And so

$$\sum_{s=0}^{\infty}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}a_s\mathrm{F}\left(-\frac{a}{2},\ \frac{a+1}{2};\ s+1;\ \frac{x_1}{e_{31}}\right)$$

$$\times \underset{x\to e_3}{\longrightarrow} \sum_{s=0}^{\infty} -\frac{\Gamma(s+\frac{1}{2})\Gamma(s+\frac{1}{2})a_s}{\Gamma\left(s+1+\frac{a}{2}\right)\Gamma\left(s+\frac{1-a}{2}\right)}[1+0(x_3)]$$

$$-\frac{2\pi a_0}{\Gamma\left(-\frac{a}{2}\right)\Gamma\left(\frac{a+1}{2}\right)}\left(\frac{x_3}{e_{13}}\right)^{1/2}[1+0(x_3)]$$

for $\left|\arg\frac{x_3}{e_{13}}\right|<\pi$. Also

$$\sum_{s=0}^{\infty}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}\alpha_s\mathrm{F}\left(-\frac{a}{2},\ \frac{a+1}{2};\ s+1;\ \frac{x_1}{e_{21}}\right)$$

$$\times \underset{z=e_3}{=} \left(\frac{e_{23}}{e_{21}}\right)^{a/2}\sum_{s=0}^{\infty}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}\alpha_s\mathrm{F}\left(-\frac{a}{2},\ s+\frac{1-a}{2};\ s+1;\ \frac{e_{31}}{e_{32}}\right).$$

For n an integer we find

$$\mathrm{A}_n=\frac{\Gamma\left(n-\frac{a}{2}\right)\Gamma\left(n+\frac{a+1}{2}\right)}{e_{21}^n\Gamma(n+\frac{1}{2})}$$

$$\times\sum_{s=0}^{n}\frac{e_{32}^s\Gamma(s+\frac{1}{2})}{\Gamma\left(s+\frac{a+1}{2}\right)\Gamma\left(s-\frac{a}{2}\right)}\frac{\mathrm{B}_s}{\Gamma(n+1-s)},$$

where B_n is derived from A_n by interchanging e_1 and e_3. And so we discover the solution

$$\sum_{n=0}^{\infty}\mathrm{A}_n x_1^n=\sum_{s=0}^{\infty}\mathrm{B}_s\left(e_{23}\frac{x_1}{x_2}\right)^s\mathrm{F}\left(-\frac{a}{2},\ \frac{a+1}{2};\ s+\tfrac{1}{2};\ \frac{x_1}{e_{21}}\right).$$

Likewise we find

$$\sum_{n=0}^{\infty}\mathrm{A}_{n+\frac{1}{2}}x_1^{n+1/2}$$

$$=\left(\frac{x_1}{e_{21}}\right)^{1/2}\sum_{s=0}^{\infty}\frac{c_s}{s+\frac{1}{2}}\left(\frac{x_1}{x_2}\right)^s\mathrm{F}\left(\frac{1-a}{2},\ 1+\frac{a}{2};\ s+\tfrac{3}{2};\ \frac{x_1}{e_{21}}\right),$$

where c_s is derived from a_s by changing e_1, e_2, e_3 into e_2, e_3, e_1. Other Lamé functions are derived from these by interchanging e_1, e_2, and e_3.

The generalized spheroidal equation is taken to be

$$(z^2-1)\frac{d^2S}{dz^2}+2(az+\alpha)\frac{dS}{dz}+\left(\frac{c^2z^2}{4}+icyz+b-\frac{c^2}{4}\right)S=0.$$

A special case of it has come from the problem of an electron in the field of two nuclei. If $S=S[a, b, c; \alpha, \gamma; z]$ denote a solution, then

$$(z-1)^{1-a-\alpha}S[1-\alpha, b+(\alpha+a-1)(\alpha-a), c; 1-a, \gamma; z],$$

$$S[a, b, -c; \alpha, -\gamma; z], \quad S[a, b, c; -\alpha, -\gamma; -z]$$

are also solutions. We need, therefore, write down but one solution of each type.

For the first type we find, by the above methods, the solution

$$S=e^{\frac{icz}{2}}\sum_{n=0}^{\infty}d_n\left(\frac{1-z}{2}\right)^n \times F\left(a-\gamma-1,\ n+a+\gamma;\ n+a+\alpha;\ \frac{1-z}{2}\right),$$

where $d_{-1}=0$ and

$$(n+1)(n+a+\alpha)d_{n+1}+2ic(n+\gamma+a)\frac{n-1+a+\gamma}{n-1+a+\alpha}d_{n-1} =[n(n+1+2\gamma+2ic)+b+(\gamma+a)(1+\gamma-a) +ic(a+\gamma+\alpha)]d_n.$$

Another type is

$$S=\sum_{n=0}^{\infty}\alpha_n(ix)^{\frac{n-a-\alpha}{2}}M_{\frac{n+a-\alpha}{2}+\gamma,\ \frac{n-1+a+\alpha}{2}},$$

where $x=c(z-1)$,

$$2ic(n+1)(n+a+\alpha)\alpha_{n+1} =[n(n-1+2a-2ic)+b+ic(\alpha-a-\gamma)]\alpha_n +\frac{(n-1+a+\gamma)(n-1+a-\alpha)}{n-1+a+\alpha}\alpha_{n-1},$$

and M is the confluent hypergeometric function in the notation of Whittaker and Watson, ' Modern Analysis.'

We find by the usual methods that, for characteristic b, this solution can be written

$$S = \sum_{n=0}^{\infty} a_n''(ix)^{-\frac{n}{2}-a} M_{\frac{n}{2}+\gamma,\ \frac{n-1}{2}+a}^{(ix)},$$

where

$$a_n = (2ic)^n \frac{\Gamma(n+a+\alpha)}{\Gamma(n+2a)} \alpha_n, \qquad \alpha_n \underset{n\to\infty}{\longrightarrow} \frac{(-1)^n[1+0(n^{-1})]}{\Gamma(n+1+2\alpha+a-\gamma)}.$$

It can also be written

$$S = \sum_{n=0}^{\infty} b_n''(-ix)^{-\frac{n}{2}-a} M_{\frac{n}{2}-\gamma,\ \frac{n-1}{2}+a}^{(-ix)},$$

where b_n is obtained from a_n by replacing c, γ with $-c$, $-\gamma$. From these two forms we have, for $|\arg x| < \pi$,

$$1 \underset{x\to 0}{\longrightarrow} S \underset{x\to\infty}{\longrightarrow} \frac{\Gamma(2a)}{\Gamma(a-\gamma)} \frac{e^{\frac{ix}{2}}[1+0(x^{-1})]}{\sum_{n=0}^{\infty} a_n''/a_0''(ix)^{a+\gamma}} + \frac{\Gamma(2a)}{\Gamma(a+\gamma)} \frac{e^{-\frac{ix}{2}}[1+0(x^{-1})]}{\sum_{n=0}^{\infty} b_n''/b_0''(-ix)^{a-\gamma}}.$$

A proof of this is given below.

Finally we notice another kind of characteristic solution. We have the formal solution

$$S = e^{\frac{icz}{2}}(z+1)^{-a-\gamma} \sum_{n=0}^{\infty} c_n \left(\frac{z+1}{2}\right)^{-n};$$

this series is, in general, divergent; but for characteristic values of b, $c_{-1}=0$. In this case we find the form

$$S = e^{\frac{icz}{2}}(z+1)^{-a-\gamma} \times \sum_{n=0}^{\infty} \gamma_n'' F\left(a+\gamma,\ 1+\alpha+\gamma;\ n+2+2\gamma;\ \frac{2}{z+1}\right),$$

where

$$(n+1)(n+2+2\gamma)\gamma_{n+1} + 2ic\frac{(n+1+\gamma-a)(n+\gamma-\alpha)}{n+1+2\gamma}\gamma_{n-1} = [n(n+1+2\gamma+2ic)+(\gamma+a)(1+\gamma-\alpha) + b + ic(2+\gamma-a-\alpha)]\gamma_n.$$

With this form we can join at once to $z=1$.

Joining proof. Denote

$$x = x_1 + ix_2, \quad a_1 = \text{real } (a), \quad \gamma_1 = \text{real } (\gamma),$$

and suppose first

$$\lim_{x=\infty} \frac{x_2}{|x|} > \delta > 0.$$

Then, if $N + a_1 - \gamma_1 > 0$,

$$\sum_{n=N}^{\infty} b_n''(-ix)^{-\frac{n}{2}-a} M_{\frac{n}{2}-\gamma,\ \frac{n-1}{2}+a}^{(-ix)}$$

$$= \frac{e^{-\frac{ix}{2}}}{\Gamma(a+\gamma)} \sum_{n=N}^{\infty} \frac{\Gamma(n+2a)b_n''}{\Gamma(n+a-\gamma)} \left(\int_0^{1/2} + \frac{1}{1-e^{2\pi i(a+\gamma)}} \int_{1/2}^{(1+)} \right) \times (1-v)^{a+\gamma-1} v^{n+a-\gamma-1} e^{ixv}\, dv.$$

$$\left| \sum_{n=N}^{\infty} \frac{\Gamma(n+2a)b_n''}{\Gamma(n+a-\gamma)} \int_0^{1/2} \right| < \sum_{n=0}^{\infty} \left| \frac{\Gamma(n+2a)b_n''}{\Gamma(n+a-\gamma)} \right| \int_0^{1/2} \times (1-v)^{a_1+\gamma_1-1} v^{N+a_1-\gamma_1-1} e^{-x_2 v}\, dv = 0(x_2^{-N-a_1+\gamma_1}).$$

For the second sum first assume $a_1 + \gamma_1 > 0$. Then

$$\left| \sum_{n=N}^{\infty} \frac{\Gamma(n+2a)b_n''}{\Gamma(n+a-\gamma)} \frac{1}{1-e^{2\pi i(a+\gamma)}} \int_{1/2}^{(1+)} \right| < \sum_{n=N}^{\infty} \left| \frac{\Gamma(n+2a)b_n''}{\Gamma(n+a-\gamma)} \right| \times \frac{\Gamma(n+a_1-\gamma_1)}{\Gamma(n+2a_1)} \Gamma(a_1+\gamma_1) e^{-\frac{x_2}{2}} = 0\left(e^{-\frac{x_2}{2}}\right).$$

The same estimate holds for $a_1 + \gamma \leqslant 0$, where first we integrate by parts until the contour integral can collapse into line integrals.

Therefore we have, for $\pi > \arg x > 0$,

$$\sum_{n=0}^{\infty} b_n'' \underset{x\to 0}{\longleftarrow} S \underset{x\to\infty}{\longrightarrow} b_0''(-ix)^{-a} M_{-\gamma,\ a-1/2}^{(-ix)}[1+0(x^{-1})] \sim \frac{\Gamma(2a)b_0''}{\Gamma(a+\gamma)}(-ix)^{-a+\gamma} e^{-\frac{ix}{2}}.$$

In the same way we find, for $-\pi < \arg x < 0$,

$$\sum_{n=0}^{\infty} a_n'' \underset{x\to 0}{\longleftarrow} S \underset{x\to\infty}{\longrightarrow} \frac{\Gamma(2a)a_0''}{\Gamma(a-\gamma)}(ix)^{-a-\gamma} e^{\frac{ix}{2}}[1+0(x^{-1})].$$

And so the above joining is proved for $\pi > |\arg x| > 0$.

Suppose now

$$\lim_{x=\infty} \left| \frac{x_1}{x} \right| > \delta > 0.$$

For simplicity we assume $a_1 \pm \gamma_1 > 0$: the general case is similarly treated, where we replace $\int_0^1$ with $\oint^{(1+,\,0+,\,1-,\,0-)}$. Then

$$\sum_{n=0}^{\infty} b_n''(-ix)^{-\frac{n}{2}-a} M_{\frac{n}{2}-\gamma,\ \frac{n-1}{2}+a}^{(-ix)}$$

$$= \frac{e^{-\frac{ix}{2}}}{\Gamma(a+\gamma)} \int_0^1 (1-v)^{a+\gamma-1} v^{a-\gamma-1} \sum_{n=0}^{\infty} \frac{\Gamma(n+2a) b_n''}{\Gamma(n+a-\gamma)} v^n e^{ixv}\, dv.$$

Take $x_1 > 0$: the case, $x_1 < 0$, is similarly treated. From the asymptotic solutions of the differential equation satisfied by

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+2a) b_n''}{\Gamma(n+a-\gamma)} v^n$$

we see that, neglecting $0(e^{-x_1})$,

$$\int_0^1 = \int_0^i + \int_{1+i}^1.$$

Also

$$\int_0^i = \Gamma(2a) b_0'' \frac{1+0(x^{-1})}{(-ix)^{a-\gamma}},$$

$$\int_{1+i}^1 = \sum_{n=0}^{\infty} \frac{\Gamma(n+2a) b_n''}{(\Gamma n+a-\gamma)} \Gamma(a+\gamma) e^{ix} (ix)^{-a-\gamma} [1+0(x^{-1})].$$

Therefore the above joining holds for $|\arg x| < \pi$ provided

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+2\,a)}{\Gamma(n+a-\gamma)} \frac{b_n''}{b_0} \Big/ \sum_{n=0}^{\infty} \frac{b_n''}{b_0''} = \frac{\Gamma(2a)}{\Gamma(a-\gamma)} \Big/ \sum_{n=0}^{\infty} \frac{a_n''}{a_0''}.$$

Now, since

$$\frac{b_n}{b_0} = \frac{\Gamma(n+a-\alpha)}{\Gamma(n+2a)} (2ic)^n \sum_{s=0}^{n} \frac{(2ic)^{-s}}{\Gamma(n+1-s)} \frac{a_s}{a_0} \frac{\Gamma(s+2a)}{\Gamma(s+a-\alpha)},$$

this condition can be written

$$e^{-ic} \sum_{n=0}^{\infty} \frac{\Gamma(n+2a)}{\Gamma(n+a-\gamma)} \frac{b_n''}{b_0''} \Big/ \frac{\Gamma(2a)}{\Gamma(a-\gamma)}$$

$$= \sum_{n=0}^{\infty} \frac{a_n''}{a_0''} (-2ic)^{-\frac{n}{2}-a} M_{\frac{n}{2}-a,\ \frac{n-1}{2}+a}^{(-2ic)} \Big/ \sum_{n=0}^{\infty} \frac{a_n''}{a_n''}.$$

And this last is true, since

$$e^{\frac{ix}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+2a)b_n''}{\Gamma(n+a-\gamma)} \left(-\frac{x}{2c}\right)^n$$

and

$$\sum_{n=0}^{\infty} a_n''(ix)^{-\frac{n}{2}-a} \mathrm{M}_{n-\frac{1}{2}-a,\ \frac{n-1}{2}+a}^{(ix)}$$

are different forms of the integral solution of

$$(z^2-1)\frac{d^2\mathrm{S}}{dz^2}+2(az-\gamma)\frac{d\mathrm{S}}{dz}+\left(\frac{c^2z^2}{4}-icxz+b-\frac{c^2}{4}\right)\mathrm{S}=0.$$

XXIV. *Transformation of Curves arising from Lebesgue Complex Integration.* *By* W. Fabian, *M.A., Ph.D.**

1. *Introduction.*

WE shall give here some geometrical deductions from Lebesgue Complex Integration. For this purpose we shall establish preliminary theorems on sets of points in two complex planes and on the change of the variable in a complex Lebesgue integral, which, being of interest in themselves, we therefore give in a general form.

2. *Sets of Points in two Complex Planes.*

Let us define bounded simple curves l and λ in an interval $(t_0 \leqslant t \leqslant \mathrm{T})$ by

$$z=x+iy=x(t)+iy(t)$$

and

$$w=u+iv=u(t)+iv(t)$$

respectively, where $x(t)$, $y(t)$, $u(t)$, and $v(t)$ are continuous monotonic one-valued functions of the real parameter t for all values of t such that $t_0 \leqslant t \leqslant \mathrm{T}$, where t_0 and T are finite; and neither $x(t)+iy(t)$ nor $u(t)+iv(t)$ assumes the same value for any two different values of t in the range $t_0 \leqslant t \leqslant \mathrm{T}$.

There is then a one-to-one correspondence between l and λ.

If $\mathrm{E}(l)$ and $\mathrm{E}(\lambda)$ denote corresponding sets of points on l and λ respectively, and $m[\mathrm{E}(l)]$ and $m[\mathrm{E}(\lambda)]$ their

* Communicated by the Author.

measures if these sets are measurable *, then the following three theorems can be seen to follow almost immediately:—

Theorem 1.—*If* $m[\mathrm{E}(l)]=0$, *then* $m[\mathrm{E}(\lambda)]=0$.

Theorem 2.—*If* $\mathrm{E}(l)$ *is measurable, then* $\mathrm{E}(\lambda)$ *is measurable.*

Theorem 3.—*If* $f(z)$ *is measurable in a measurable set of points* $\mathrm{E}(l)$, *then* $f(z)$ *is measurable in* $\mathrm{E}(\lambda)$.

Theorem 1 follows from the fact that, under the hypothesis of this theorem, the points composing $\mathrm{E}(l)$ and $\mathrm{E}(\lambda)$ are finite in number or enumerably infinite. Theorem 2 then follows from this theorem, and the first two theorems which we established in our work on Lebesgue Complex Integration †.

Proof of Theorem 3.

Putting $f(z)=\mathrm{P}+i\mathrm{Q}$, we observe. under the hypothesis of this theorem, that the set of points in $\mathrm{E}(l)$, for which $\mathrm{P}>\mathrm{A}$, is measurable for every constant A. Hence, by theorem 2, the set of points in $\mathrm{E}(\lambda)$, for which $\mathrm{P}>\mathrm{A}$, is measurable for every constant A, so that P is measurable in $\mathrm{E}(\lambda)$. Similarly Q, and hence $f(z)$, is measurable in $\mathrm{E}(\lambda)$.

3. *Change of Variable in a Complex Lebesgue Integral.*

Let E_t be the set of points on the t-segment corresponding to $\mathrm{E}(l)$. The Lebesgue integral of $f(z)=\mathrm{P}+i\mathrm{Q}$, where P and Q are real, taken over $\mathrm{E}(l)$, is defined by ‡

$$\int_{\mathrm{E}(l)} f(z)\,dz=\int_{\mathrm{E}_t}\mathrm{P}\,dx\,(t)-\int_{\mathrm{E}_t}\mathrm{Q}\,dy\,(t)+i\int_{\mathrm{E}_t}\mathrm{Q}\,dx\,(t)+i\int_{\mathrm{E}_t}\mathrm{P}\,dy\,(t),$$

if each of these Lebesgue-Stieltjes integrals, taken over E_t, exists.

Theorem 4.—*If a function* $f(z)$ *is measurable in, and integrable over, a measurable set of points* $\mathrm{E}(l)$, *then*

$$\int_{\mathrm{E}(l)} f(z)\,dz=\int_{\mathrm{E}(\lambda)} f(z)\frac{dz}{dw}\,dw.$$

* Fabian, Phil. Mag. ser. 7, vol. xxii. pp. 523–534 (1936). Measure and measurable functions are discussed here.

† Fabian, Phil. Mag. ser. 7, vol. xxii. pp. 524, 525 (1936).

‡ Fabian, Phil. Mag. ser. 7, vol. xxii. pp. 523–534, 1117–1122 (1936).

Proof.—Let E_s and E_σ be the sets of points corresponding to $E(l)$ and $E(\lambda)$ on the segments $(0 \leqslant s \leqslant S)$ and $(O \leqslant \sigma \leqslant \sigma_1)$ respectively, where s and σ denote the lengths of those arcs of l and λ respectively which correspond to the interval (t_0, t), and $s=S$, $\sigma=\sigma_1$, when $t=T$. Let CE_s be the set of points complementary to E_s in the interval $(O \leqslant s \leqslant S)$. Let $f_1(z)=P_1+iQ_1$, where P_1 and Q_1 are real, be a function which is equal to $f(z)=P+iQ$ at all points of E_s and zero at all points of CE_s.

By a previous theorem *,

$$\int_{E(l)} f(z)\,dz = \int_{E_s} f(z)\frac{dz}{ds}\,ds \quad . \quad . \quad . \quad . \quad (1)$$

$$= \int_0^S f_1(z)\frac{dz}{ds}\,ds$$

$$= \int_0^S \left(P_1\frac{dx}{ds} - Q_1\frac{dy}{ds}\right)ds + i\int_0^S \left(Q_1\frac{dx}{ds} + P_1\frac{dy}{ds}\right)ds.$$

Since there corresponds to every set of points of measure zero in $(O \leqslant \sigma \leqslant \sigma_1)$ a set of points of measure zero in $(O \leqslant s \leqslant S)$, and s is a continuous monotonic function of σ in $(O \leqslant \sigma \leqslant \sigma_1)$, therefore †

$$s = \int_0^\sigma \frac{ds}{d\sigma}\,d\sigma \text{ in } (O \leqslant \sigma \leqslant \sigma_1).$$

Hence ‡

$$\int_0^S f_1(z)\frac{dz}{ds}\,ds = \int_0^{\sigma_1} f_1(z)\frac{dz}{ds}\cdot\frac{ds}{d\sigma}\,d\sigma.$$

Therefore

$$\int_{E_s} f(z)\frac{dz}{ds}\,ds = \int_{E_\sigma} f(z)\frac{dz}{ds}\cdot\frac{ds}{d\sigma}\,d\sigma$$

$$= \int_{E_\sigma} f(z)\frac{dz}{d\sigma}\,d\sigma$$

$$= \int_{E\sigma} f(z)\frac{dz}{dw}\cdot\frac{dw}{d\sigma}\,d\sigma$$

$$= \int_{E(\lambda)} f(z)\frac{dz}{dw}\,dw, \quad . \quad . \quad . \quad (2)$$

* Fabian, Phil. Mag. ser. 7, vol. xxii. pp. 523–534 (1936).

† Hobson, ' Theory of Functions of a Real Variable,' vol. i. p. 547 (1921).

‡ Hobson, ' Theory of Functions of a Real Variable,' vol. i. p. 592 (1921).

by theorem 5 of our last paper on this subject *. By (1) and (2) the conclusion follows.

Hence we obtain the following converse to this theorem :—

Theorem 5.—*If* $\int_{E(\lambda)} f(z)\frac{dz}{dw}dw$, *where* $f(z)$ *is measurable in the measurable set of points* $E(\lambda)$, *exists as a finite number, then*

$$\int_{E(\lambda)} f(z)\frac{dz}{dw}dw = \int_{E(l)} f(z)dz.$$

For by theorem 4

$$\int_{E(\lambda)} f(z)\frac{dz}{dw}dw = \int_{E(l)} f(z)\frac{dz}{dw}\cdot\frac{dw}{dz}dz$$
$$= \int_{E(l)} f(z)\,dz.$$

4. *Transformation of Curves.*

Theorem 6.—*Suppose that the indefinite integral*

$$F(z)=F(z_0)+\int_{z_0}^{z} f(z)dz,$$

taken along l, *exists everywhere on* l, *where* $f(z)$ *is a one-valued measurable function of* z *on* l. *Suppose that, almost everywhere in* (t_0, T),

$$\theta_1(t)\equiv P\frac{dx}{dt} - Q\frac{dy}{dt} \geqslant 0,$$

$$\theta_2(t)\equiv Q\frac{dx}{dt} + P\frac{dy}{dt} \geqslant 0,$$

and

$$\theta_1(t)+i\theta_2(t)\neq 0,$$

where P *and* Q *are the real and imaginary parts of* $f(z)$ *respectively.*

Then l *is transformed by the relation* $w=F(z)$ *into a curve* λ †.

Proof.—By Theorem 4,

$$F(z)=F(z_0)+\int_{t_0}^{t} f(z)\frac{dz}{dt}dt$$
$$= P_0+\int_{t_0}^{t}\theta_1(t)\,dt+i\{Q_0+\int_{t_0}^{}\theta_2(t)\,dt\},$$

* Fabian, Phil. Mag. ser. 7, vol xxii. pp. 1117–1122 (1936).
† As defined in § 2 in this paper.

where P_0 and Q_0 are the real and imaginary parts of $F(z_0)$ respectively. Hence it follows from the hypothesis that, for the relation $u+iv=w=F(z)$, u and v are bounded continuous monotonic one-valued functions of t in (t_0, T).

To show that $F(z)$ does not assume the same value for any two different values of t in $(t_0 \leqslant t \leqslant T)$, consider an interval δt in (t_0, T), in which $\theta_1(t)$ is zero almost everywhere. Since, by hypothesis, $\theta_1(t)+i\theta_2(t)$ is non-zero, and $\theta_2(t) \geqslant 0$, almost everywhere in (t_0, T), therefore $F(z)$ does not assume the same value for any two different values of t in δt. Hence it follows, since by hypothesis $\theta_1(t) \geqslant 0$ almost everywhere in (t_0, T), that $F(z)$ does not assume the same value for any two different values of t in $(t_0 \leqslant t \leqslant T)$.

The theorem is proved.

In a similar way we obtain the following theorem (where the same notation is used as in the last theorem) :—

Theorem 7.—Suppose that the indefinite integral

$$F(z)=F(z_0)+\int_{z_0}^{z} f(z)\,dz,$$

taken along l, exists everywhere on l, where $f(z)$ is a one-valued measurable function of z on l. Suppose that, almost everywhere in (t_0, T),

$$\theta_1(t)+i\theta_2(t) \neq 0,$$

and either (1): $\theta_1(t) \leqslant 0, \quad \theta_2(t) \geqslant 0$;

or (2): $\theta_1(t) \geqslant 0, \quad \theta_2(t) \leqslant 0$;

or (3): $\theta_1(t) \leqslant 0, \quad \theta_2(t) \leqslant 0$.

Then l is transformed by the relation $w=F(z)$ into a curve λ.

Theorem 8.—Suppose that the measurable function $f(z)$ is uniform, real and either $\geqslant 0$ or $\leqslant 0$ almost everywhere on l, and that the zeros of $f(z)$ on l, if any, form a set of points of measure zero.

If the indefinite integral

$$F(z)=F(z_0)+\int_{z_0}^{z} f(z)\,dz,$$

taken along l, exists everywhere on l, then the relation $w=F(z)$ transforms l into a curve λ.

Proof.—Since $x(t)$ and $y(t)$ are monotonic functions of t in (t_0, T), therefore $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are either $\geqslant \mathrm{O}$ or $\leqslant \mathrm{O}$ almost everywhere in (t_0, T). Hence it follows from the hypothesis that

$$f(z)\frac{dx}{dt} \geqslant \mathrm{O} \text{ or } \leqslant \mathrm{O},$$

and

$$f(z)\frac{dy}{dt} \geqslant \mathrm{O} \text{ or } \leqslant \mathrm{O}$$

almost everywhere in (t_0, T). It will now be seen that we shall be able to deduce the conclusion from theorems 6 and 7 if we show that $\frac{dz}{dt}$ is non-zero almost everywhere in (t_0, T). To show this we observe that

$$\frac{dz}{dt} = \frac{dz}{ds} \cdot \frac{ds}{dt}, \quad \ldots\ldots \quad (1)$$

where s denotes the length of that arc of l which corresponds to the interval (t_0, t) and $s=\mathrm{S}$ when $t=\mathrm{T}$. Since $x(t)$ and $y(t)$ are monotonic functions of s in (O, S), therefore $\frac{dx}{ds}$ and $\frac{dy}{ds}$ exist as finite numbers almost everywhere in (O, S). Whenever these differential coefficients exist

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1.$$

Hence $\frac{dz}{ds}$ is never zero in (t_0, T). Since s is a monotonic function of t, and t a monotonic function of s, in (t_0, T) and (O, S), therefore $\frac{ds}{dt}$ and $\frac{dt}{ds}$ exist as finite numbers almost everywhere in (t_0, T), and the zeros of $\frac{ds}{dt}$ in (t_0, T), if any, form consequently a set of points of measure zero. $\frac{dz}{dt}$ thus exists, by (1), almost everywhere in

(t_0, T) as a finite non-zero number. The conclusion follows now from theorems 6 and 7.

Theorem 9.—*Let the measurable function* $f(z)=iQ$ *be such that, almost everywhere on* l, Q *is uniform, non-zero, and either* $\geqslant 0$ *or* $\leqslant 0$. *If the indefinite integral*

$$F(z)=F(z_0)+\int_{z_0}^{z} f(z)\,dz,$$

taken along l, *exists everywhere on* l, *then the relation* $w=F(z)$ *transforms* l *into a curve* λ.

Proof.—We can show, as in the proof of theorem 8, that

$$\frac{dz}{dt}=\frac{dx}{dt}+i\frac{dy}{dt}$$

exists as a finite non-zero number almost everywhere in (t_0, T). Therefore $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are not both zero for the same values of t in (t_0, T) except possibly in a set of points of measure zero. Consequently

$$iQ\frac{dx}{dt}-Q\frac{dy}{dt}$$

is non-zero almost everywhere in (t_0, T). It also follows from the hypothesis, since $x(t)$ and $y(t)$ are monotonic in (t_0, T), that

$$Q\frac{dx}{dt}\geqslant 0 \text{ or } \leqslant 0$$

and

$$Q\frac{dy}{dt}\geqslant 0 \text{ or } \leqslant 0$$

almost everywhere in (t_0, T). Hence the conclusion follows from theorems 6 and 7.

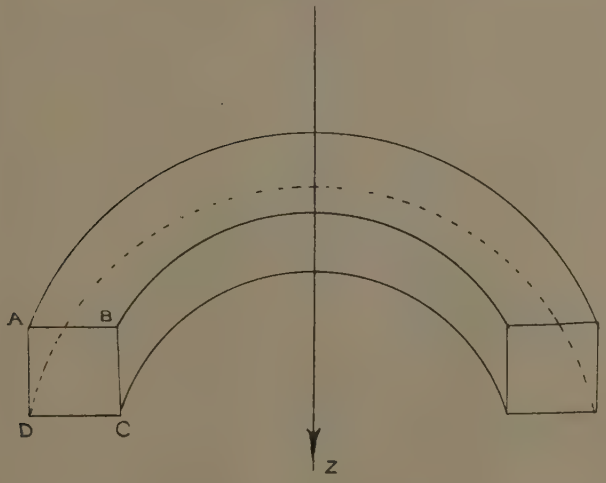
XXV. *Note on the Torsion of a Curved Rod.*

By Bibhutibhusan Sen, *Krishnagar, Bengal* *.

1. *Introduction.*

THE curved rod discussed in this paper consists of a sector of a circular ring formed by the surfaces $r=a$, $r=b$, and $z=\pm c$, the equations being expressed in cylindrical coordinates (fig. 1). A method of successive approximations has been used by O. Göhner † in solving

Fig. 1.



the torsion problems of curved rods having different forms of cross-sections. In the present note an exact solution of the torsion problem for a curved rod having the same form as mentioned above has been obtained in terms of a convergent series. It is found that the numerical calculation up to the required degree of accuracy can be made by taking only a few terms of this series.

* Communicated by the Author.

† O. Göhner, 'Ingenieur-Archiv,' vol. i. p. 619 (1930), and vol. ii. pp. 1 & 381 (1931). The solution for the ring sector of circular section has been reproduced by Prof. S. Timoshenko in his 'Theory of Elasticity,' p. 356 (1934).

Assuming that all stress components excepting $\widehat{r\theta}$ and $\widehat{\theta z}$ are zero, we get the equation of equilibrium in the form

$$\frac{\partial \widehat{r\theta}}{\partial r} + \frac{\partial \widehat{\theta z}}{\partial z} + 2\frac{\widehat{r\theta}}{r} = 0, \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

and the two compatibility equations as

$$\left.\begin{aligned} \frac{\partial^2 \widehat{\theta z}}{\partial r^2} + \frac{1}{r}\frac{\partial \widehat{\theta z}}{\partial r} + \frac{\partial^2 \widehat{\theta z}}{\partial z^2} - \frac{\widehat{\theta z}}{r^2} &= 0, \\ \frac{\partial^2 \widehat{r\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial \widehat{r\theta}}{\partial r} + \frac{\partial^2 \widehat{r\theta}}{\partial z^2} - \frac{4\widehat{r\theta}}{r^2} &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad (1.2)$$

We satisfy the equation (1.1) by taking

$$\widehat{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial z}, \quad \widehat{\theta z} = -\frac{1}{r^2}\frac{\partial \phi}{\partial r}. \quad . \quad . \quad . \quad . \quad (1.3)$$

Substituting these values in (1.2), we obtain

$$\left.\begin{aligned} \frac{\partial}{\partial r}\left[\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{3}{r}\frac{\partial \phi}{\partial r}\right] &= 0, \\ \frac{\partial}{\partial z}\left[\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{3}{r}\frac{\partial \phi}{\partial r}\right] &= 0. \end{aligned}\right\} \quad . \quad . \quad (1.4)$$

Hence we find that

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z} - \frac{3}{r}\frac{\partial \phi}{\partial r} = \text{constant}.$$

Let us put this constant$=2\mathrm{T}$, ... (1.5)

and write

$$\phi = \mathrm{T}(z^2 - c^2) + \phi_1. \quad . \quad . \quad . \quad . \quad (1.6)$$

Then ϕ_1 will satisfy the equation

$$\frac{\partial^2 \phi_1}{\partial r^2} + \frac{\partial^2 \phi_1}{\partial z_1^2} - \frac{3}{r}\frac{\partial \phi_1}{\partial r} = 0. \quad . \quad . \quad . \quad (1.7)$$

As a solution of the above equation let us write

$$\phi_1 = r^2 \phi_2 \cos \nu_n z, \quad . \quad . \quad . \quad . \quad . \quad (1.8)$$

in which ϕ_2 is a function of r only and

$$\nu_n = \frac{2n+1}{2c}\pi, \quad . \quad . \quad . \quad . \quad . \quad (1.9)$$

n being an integer.

Substituting (1.8) in (1.7), we get the equation satisfied by ϕ_2 as

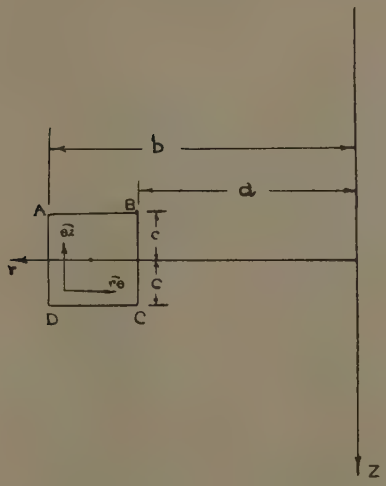
$$\frac{\partial^2\phi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_2}{\partial r}-\left(\nu_n^2+\frac{4}{r^2}\right)\phi_2=0. \quad . \quad . \quad (1.10)$$

The solution of this equation is

$$\phi_2=A_nI_2(\nu_n r)+B_nK_2(\nu_n r), \quad . \quad . \quad . \quad (1.11)$$

where A_n and B_n are constants and I_2, K_2 are the second order Bessel functions of the first and second kinds respectively having imaginary arguments.

Fig. 2.



Hence we can write

$$\phi=T(z^2-c^2)+\sum_{n=0}^{\infty} r^2[A_nI_2(\nu_n r)+B_nK_2(\nu_n r)]\cos\nu_n z. \quad (1.12)$$

If the lateral surfaces be free from stress we must have

$$\frac{1}{r^2}\frac{\partial\phi}{\partial r}=0 \quad \text{when } z=\pm c, \quad . \quad . \quad . \quad . \quad (1.13)$$

$$\frac{1}{r^2}\frac{\partial\phi}{\partial z}=0 \quad \text{when } r=a \text{ and } r=b. \quad . \quad (1.14)$$

In the problem considered we have taken $b>r>a$ (fig. 2).

Since $\cos \nu_n z=0$ when $z=\pm c$, we find that the condition (1.3) is identically satisfied.

Again expressed in Fourier Series

$$T(c^2-z^2)=\frac{32c^2}{\pi^3}\tau \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos \nu_n z, \quad . \quad (1.15)$$

the expansion being valid for

$$c \geqslant z \geqslant -c.$$

Then putting

$$L_n=\frac{32c^2}{\pi^3}\frac{(-1)^n}{(2n+1)^3}, \quad . \quad . \quad . \quad . \quad (1.16)$$

we can express (1.12) in the alternative form as

$$\phi=\sum_{n=0}^{\infty}[r^2\{A_n I_2(\nu_n r)+B_n K_2(\nu_n r)\}-TL_n]\cos \nu_n z. \quad (1.17)$$

Hence the boundary conditions (1.14) will be satisfied if

$$\left.\begin{aligned} A_n a^2 I_2(\nu_n a)+B_n a^2 K_2(\nu_n a)&=TL_n,\\ A_n b^2 I_2(\nu_n b)+B_n b^2 K_2(\nu_n b)&=TL_n,\end{aligned}\right\} \quad . \quad (1.18)$$

from which we obtain

$$A_n=\frac{TL_n}{a^2 I_2(\nu_n b)}\frac{\left[\dfrac{a^2}{b^2}-\dfrac{K_2(\nu_n b)}{K_2(\nu_n a)}\right]}{\left[1-\dfrac{I_2(\nu_n a)}{I_2(\nu_n b)}\cdot\dfrac{K_2(\nu_n b)}{K_2(\nu_n a)}\right]}, \quad . \quad (1.19)$$

$$B_n=\frac{TL_n}{a^2 K_2(\nu_n a)}\frac{\left[1-\dfrac{a^2}{b^2}\dfrac{I_2(\nu_n a)}{I_2(\nu_n b)}\right]}{\left[1-\dfrac{I_2(\nu_n a)}{I_2(\nu_n b)}\cdot\dfrac{K_2(\nu_n b)}{K_2(\nu_n a)}\right]}. \quad . \quad (1.20)$$

2. *Convergency of the Series for* ϕ.

The series

$$\sum_{n=0}^{\infty} A_n I_2(\nu_n r)\cos \nu_n z \quad . \quad . \quad . \quad . \quad (2.1)$$

will be certainly convergent if

$$\sum_{n=0}^{\infty}|A_n I_2(\nu_n r)| \text{ is convergent.}$$

We know that *

$$\text{Lt } \nu_n \to \infty \ \frac{K_2(\nu_n b)}{K_2(\nu_n a)} = \text{Lt } \nu_n \to \infty \ \frac{\sqrt{\frac{\pi}{2\nu_n b}}\, e^{-\nu_n b}}{\sqrt{\frac{\pi}{2\nu_n a}}\, e^{-\nu_n a}}$$

$$= \text{Lt } \nu_n \to \infty \ \sqrt{\frac{a}{b}}\, e^{-\nu_n (b-a)}$$

$$= 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

and

$$\text{Lt } \nu_n \to \infty \ \frac{I_2(\nu_n a)}{I_2(\nu_n b)}$$

$$= \text{Lt } \nu_n \to \infty \ \frac{\frac{1}{\sqrt{2\pi\nu_n a}}\, e^{\nu_n a}[1 + \text{a small quantity}]}{\frac{1}{\sqrt{2\pi\nu_n b}}\, e^{\nu_n b}[1 + \text{a small quantity}]}$$

$$= \text{Lt } \nu_n \to \infty \ \sqrt{\frac{b}{a}}\, e^{-\nu_n (b-a)}$$

$$= 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

Hence writing $|U_n|$ for the nth term of the series $\sum_{n=0}^{\infty} |A_n I_2(\nu_n r)|$ we get

$$\text{Lt } \frac{|U_{n+1}|}{|U_n|} = \text{Lt } n \to \infty \ \frac{\frac{I_2(\nu_{n+1} r)}{I_2(\nu_{n+1} b)}}{\frac{I_2(\nu_n r)}{I_2(\nu_n b)}}$$

$$= \text{Lt } n \to \infty \ e^{-(\nu_{n+1} - \nu_n)(b-r)}.$$

As we have assumed $b > r > a$, the right-hand side of the above is less than unity, and thus the simple ratio test shows that the series

$$\sum_{n=0}^{\infty} |A_n I_2(\nu_n r)|,$$

* *Vide* 'Treatise on Bessel Functions,' by MacRobert, Gray, and Mathews, second edition, p. 61.

and consequently the series

$$\sum_{n=0}^{\infty} A_n I_2(\nu_n r) \cos \nu_n z,$$

are convergent.

Again putting

$$\Sigma | v_n | = \Sigma | B_n K_2(\nu_n r) |,$$

we have

$$\mathrm{Lt} \left| \frac{v_{n+1}}{v_n} \right| = \mathrm{Lt}\, n \to \infty \frac{\dfrac{K_2(\nu_{n+1} r)}{K_2(\nu_{n+1} a)}}{\dfrac{K_2(\nu_n r)}{K_2(\nu_n a)}}$$

$$= \mathrm{Lt}\, n \to \infty\, e^{-(\nu_{n+1} - \nu_n)(r-a)}.$$

The last result being less than unity

$$\sum_{n=0}^{\infty} | B_n K_2(\nu_n r) |$$

is convergent, and hence

$$\sum_{n=0}^{\infty} B_n K_2(\nu_n r) \cos \nu_n z \quad . \quad . \quad . \quad . \quad (2.4)$$

is also convergent.

Combining the two series (2.1) and (2.4) we find that the expression for ϕ is convergent.

3. *Expression for the Torque.*

Taking moments of the stresses $\widehat{\theta z}$ and $\widehat{r\theta}$ about the centre of a meridian section (fig. 2), we have the torque

$$Q_t = \int_a^b \int_{-c}^{c} \left[\frac{z}{r^2} \frac{\partial \phi}{\partial z} + \left(r - \frac{a+b}{2} \right) \frac{1}{r^2} \frac{\partial \phi}{\partial r} \right] dr\, dz. \quad (3.1)$$

If we split up this integral into three parts we find, from (1.12),

$$\vartheta_1 = \int_b^b \int_{-c}^{c} \frac{z}{r^2} \frac{\partial \phi}{\partial z} dr\, dz$$

$$= \int_a^b \int_{-c}^{c} \frac{1}{r^2} \cdot 2\mathrm{T} z^2 dr dz$$

$$+ \sum_{n=0}^{\infty} \int_a^b \int_{-c}^{c} \left[\{ A_n I_2(\nu_n r) + B_n K_2(\nu_n r) \} \right.$$

$$\times z\frac{d}{dz}\cos\frac{2n+1}{2c}\pi z\Big]\,dr\,dz$$

$$=\frac{4\mathrm{T}c^3}{3}\frac{(b-a)}{ab}+\sum_{n=0}^{\infty}-\frac{4c(-1)^n}{(2n+1)\pi}$$

$$\times\int_a^b[\mathrm{A}_n\mathrm{I}_2(\nu_n r)+\mathrm{B}_n\mathrm{K}_2(\nu_n r)]dr. \quad . \quad (3.2)$$

$$\mathfrak{S}_2=\int_a^b\int_{-c}^{c}\frac{1}{r}\frac{\partial\phi}{\partial r}\,dr\,dz$$

$$=\sum_{n=0}^{\infty}\int_a^b\int_{-c}^{c}\frac{1}{r}\frac{d}{dr}[r^2\{\mathrm{A}_n\mathrm{I}_2(\nu_n r)+\mathrm{B}_n\mathrm{K}_2(\nu_n r)\}]$$

$$\times\cos\frac{2n+1}{2c}\pi z\,dr\,dz$$

$$=\sum_{n=0}^{\infty}\frac{4c(-1)^n}{(2n+1)\pi}\int_a^b\frac{1}{r}\frac{d}{dr}[r^2\{\mathrm{A}_n\mathrm{I}_2(\nu_n r)+\mathrm{B}_n\mathrm{K}_2(\nu_n r)\}]\,dr$$

$$=\sum_{n=0}^{\infty}\frac{4c(-1)^n}{(2n+1)\pi}\Big[r\{\mathrm{A}_n\mathrm{I}_2(\nu_n r)+\mathrm{B}_n\mathrm{K}_2(\nu_n r)\}\Big]_a^b$$

$$+\sum_{n=0}^{\infty}\frac{4c(-1)^n}{(2n+1)\pi}\int_a^b[\mathrm{A}_n\mathrm{I}_2(\nu_n r)+\mathrm{B}_n\mathrm{K}_2(\nu_n r)]\,dr. \quad (3.3)$$

Applying the result (1.18), we have the first part of the above expression

$$=\sum_{n=0}^{\infty}\frac{4c(-1)^n}{(2n+1)\pi}\mathrm{L}_n\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$=-\frac{128c^3\mathrm{T}}{\pi^4}\frac{(b-a)}{ab}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}$$

$$=-\frac{4c^3\mathrm{T}}{3}\frac{b-a}{ab},$$

since

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}=\frac{\pi^4}{96}.$$

Thus we see that

$$\mathfrak{S}_1+\mathfrak{S}_2=0.$$

Hence

$$\mathrm{Q}_t=-\frac{a+b}{2}\int_a^b\int_{-c}^{c}\frac{1}{r^2}\frac{\partial\phi}{\partial r}\,dr\,dz$$

$$=-\frac{a+b}{2}\sum_{n=0}^{\infty}\frac{4c(-1)^n}{(2n+1)\pi}$$

$$\times\int_a^b\frac{1}{r^2}\frac{d}{dr}\{r^2[\mathrm{A}_n\mathrm{I}_2(\nu_n r)+\mathrm{B}_n\mathrm{K}_2(\nu_n r)]\}dr. \quad (3.4)$$

From the formulæ * satisfied by Bessel functions we have

$$\begin{aligned}&\frac{1}{r^2}\frac{d}{dr}[r^2\mathrm{I}_2(\nu_n r)]\\&=\nu_n\left[\mathrm{I}_2'(\nu_n r)+\frac{2}{\nu_n r}\mathrm{I}_2(\nu_n r)\right]\\&=\nu_n\mathrm{I}_1(\nu_n r)\\&=\frac{d}{dr}\mathrm{I}_0(\nu_n r),\quad\ldots\ldots\ldots\quad(3.5)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{r^2}\frac{d}{dr}[r^2\mathrm{K}_2(\nu_n r)]\\&=\nu_n\left[\mathrm{K}_2'(\nu_n r)+\frac{2}{\nu_n r}\mathrm{K}_2(\nu_n r)\right]\\&=-\nu_n\mathrm{K}_1(\nu_n r)\\&=\frac{d}{dr}\mathrm{K}_0(\nu_n r).\quad\ldots\ldots\ldots\quad(3.6)\end{aligned}$$

The results (3.5) and (3.6) make

$$\begin{aligned}\mathrm{Q}_t=-\frac{a+b}{2}\sum_{n=0}^{\infty}\frac{4c(-1)^n}{(2n+1)\pi}\Bigg[&\mathrm{A}_n\mathrm{I}_0(\nu_n b)\left\{1-\frac{\mathrm{I}_0(\nu_n a)}{\mathrm{I}_0(\nu_n b)}\right\}\\&+\mathrm{B}_n\mathrm{K}_0(\nu_n a)\left\{\frac{\mathrm{K}_0(\nu_n b)}{\mathrm{K}_0(\nu_n a)}-1\right\}\Bigg].\quad(3.7)\end{aligned}$$

From the results (1.19), (1.20), (2.2), and (2.3) it is found that as n tends to infinity the nth term of the series on the right-hand side of (3.7) bears a constant ratio to $\frac{1}{n^4}$. Then, by comparison with the series $\sum_{n=0}^{\infty}\frac{1}{n^4}$, it can be shown that the series in (3.7) is absolutely convergent. Let us now put

$$\mathrm{Q}_t=\mathrm{T.S},\quad\ldots\ldots\quad(3.8)$$

in which S stands for the convergent series

$$\begin{aligned}&\frac{64c^3(a+b)}{a^2\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\cdot\frac{1}{\left[1-\dfrac{\mathrm{I}_2(\nu_n a)}{\mathrm{I}_2(\nu_n b)}\cdot\dfrac{\mathrm{K}_2(\nu_n b)}{\mathrm{K}_2(\nu_n a)}\right]}\\&\times\left[\frac{\mathrm{I}_0(\nu_n b)}{(\mathrm{I}_2\nu_n b)}\cdot\left\{1-\frac{\mathrm{I}_0(\nu_n a)}{\mathrm{I}_0(\nu_n b)}\right\}\left\{\frac{\mathrm{K}_2(\nu_n b)}{\mathrm{K}_2(\nu_n a)}-\frac{a^2}{b^2}\right\}\right.\end{aligned}$$

* *Vide* MacRobert, Gray, and Mathews, *loc. cit.* pp. 20 & 22.

$$+\frac{\mathrm{K}_0(\nu_n a)}{\mathrm{K}_2(\nu_n a)}\left\{1-\frac{\mathrm{K}_0(\nu_n b)}{\mathrm{K}_2(\nu_n a)}\right\}\left\{1-\frac{a^2\,\mathrm{I}_2(\nu_n a)}{b^2\,\mathrm{I}_2(\nu_n b)}\right\}\Bigg]. \quad \ldots \quad (3.9)$$

S being calculated from the above series, T can be found from (3.8) in terms of Q_t, which is supposed to be known. From the results (1.19) and (1.20) it is apparent that the stress function ϕ in (1.12) can be expressed in terms of T, and hence in terms of Q_t. Thus, assuming that Q_t is known, we can find from (1.3) the stresses in the curved rod.

4. *Approximation.*

The presence of the factor $1/(2n+1)^3$ in A_n and B_n and of $1/(2n+1)^4$ in the series for S makes the process of approximation easy, as the other factor of each term in the series for ϕ and S remains small for large values of b compared with those of a and c. If we denote by $(\widehat{\theta z})_i$ the stress at the point $r=a$, $z=0$, we have

$$(\widehat{\theta z})_i=\left(-\frac{1}{r^2}\frac{\partial\phi}{\partial r}\right)_{r=a,\,z=0}$$

$$=-\sum_{n=0}^{\infty}\nu_n[\mathrm{A}_n\mathrm{I}_1(\nu_n a)-\mathrm{B}_n\mathrm{K}_1(\nu_n a)]$$

$$=\frac{16\mathrm{T}c}{a^2\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left[1-\frac{\mathrm{I}_2(\nu_n a)\mathrm{K}_2(\nu_n b)}{\mathrm{I}_2(\nu_n b)\mathrm{K}_2(\nu_n a)}\right]^{-1}$$

$$\times\left[\frac{\mathrm{I}_1(\nu_n a)}{\mathrm{I}_2(\nu_n b)}\left\{\frac{\mathrm{K}_2(\nu_n b)}{\mathrm{K}_2(\nu_n a)}-\frac{a^2}{b^2}\right\}\right.$$

$$\left.+\frac{\mathrm{K}_1(\nu_n a)}{\mathrm{K}_2(\nu_n a)}\left\{1-\frac{a^2\,\mathrm{I}_2(\nu_n a)}{b^2\,\mathrm{I}_2(\nu_n b)}\right\}\right]. \quad (4.1)$$

As an example let us take

$$b=2a=\frac{8c}{\pi}.$$

Substituting the values of several Bessel functions from the tables we get

$$\mathrm{S}=\cdot 134a^2 \text{ approximately,}$$

whence we get

$$T=7\cdot46\frac{Q_t}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

Then from (4.1) we have the approximate value of

$$(\theta z)_t=4\cdot70\frac{Q_t}{a^3}. \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

Proceeding in the same way we can obtain the values $\widehat{r\theta}$ and $\widehat{\theta z}$ at any point of the rod in terms of Q_t for different values of a, b, and c.

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Bengal.

XXVI. *On Free Vibrations with Non-Linear Elasticity.* *By* A. Gemant *.

1. During calculations resulting from an investigation on the internal friction of solids †, the author had to consider a certain type of free, non-linear vibrations, namely, those in which the restoring force is proportional to a power of the displacement different from unity. There exists a generally applicable method to compute the shape as well as the natural frequency of free, non-linear vibrations ‡, but the author found some simple and interesting relations for the type of non-linearity just mentioned which he thought worth recording.

In the following pages, therefore, these relations will be outlined briefly, together with some graphs showing the type of motion to be expected. As a physical application of the derived formulæ, the classical (non-quantum-mechanical) energy distribution of a large number of molecular oscillators will be derived, if the oscillations are supposed to be controlled by such non-linear restoring force.

* Communicated from the Engineering Laboratory, Oxford University.

† A. Gemant and W. Jackson, "The Measurement of Internal Friction of Solid Dielectric Materials," Phil. Mag. xxiii. p. 960 (1937).

‡ J. P. den Hartog, 'Mechanical Vibrations,' McGraw Hill, New York, 1934, p. 351.

2. The restoring force should be written as

$$-fx^{2n-1},$$

where x=displacement, f and n=constants. The differential equation is then

$$\frac{d^2x}{dt^2}=-a^2x^{2n-1}, \quad \ldots\ldots \quad (1)$$

if $a^2=f/m$ and m=mass of the vibrator. In multiplying both sides of (1) by twice the velocity $v=dx/dt$, one obtains

$$v^2=-\frac{a^2x^{2n}}{n}+\mathrm{C}, \quad \ldots\ldots \quad (2)$$

where C=integration constant, being the square of the maximum velocity (for $x=0$). From (2) we have

$$\frac{dx}{\sqrt{\mathrm{C}}\sqrt{1-\frac{a^2x^{2n}}{\mathrm{C}n}}}=dt. \quad \ldots\ldots \quad (3)$$

Now (3) does not generally allow an integration in finite form, but by introducing suitable variables it might be brought into a more universal form. We put

$$y=\sqrt[2n]{\frac{a^2}{\mathrm{C}n}}\,x \quad \ldots\ldots \quad (4)$$

and

$$t'=\sqrt[2n]{\frac{a^2\mathrm{C}^{n-1}}{n}}\,t, \quad \ldots\ldots \quad (5)$$

and have then, instead of (3),

$$\frac{dy}{\sqrt{1-y^{2n}}}=dt'. \quad \ldots\ldots \quad (6)$$

Here y is a periodic function of t', independently of the value of n. For $n=1$ the periodic function is the simple harmonic one. The interval of t' during which y rises from zero to unity evidently corresponds to one-quarter of the total period. Let us denote

$$\int_0^1\frac{dy}{\sqrt{1-y^{2n}}}=\frac{\pi}{2}b, \quad \ldots\ldots \quad (7)$$

where b is a function of n. Let us further denote

$$t'=\omega t, \quad \ldots\ldots \quad (8)$$

in analogy to the simple harmonic case. From (8) and (5) we have for ω

$$\omega = \sqrt[2n]{\frac{a^2 C^{n-1}}{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

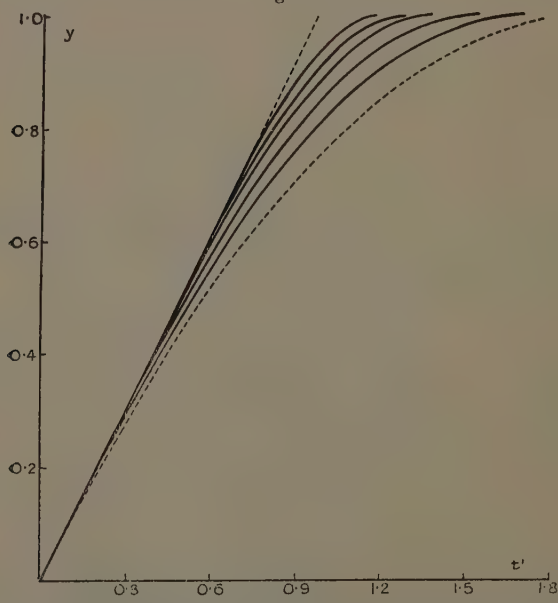
Now for a quarter-period t must assume the value $\tau/4$, if τ is the duration of one cycle. Thus, from (7) and (8)

$$\frac{\pi}{2} b = \omega \frac{\tau}{4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or

$$\omega = 2\pi b \nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Fig. 1.



Quarter oscillations for $2n$ (from left): ∞, 6, 4, 3, 2, 1·5, 1.

where $\nu = 1/\tau$ = frequency of the free vibrations. (For $n=1$, evidently $b=1$.)

3. The next step is to compute the curves $y(t')$ for different n and to find especially the curve $b(n)$, which is important for the evaluation of the natural frequency (by means of (11)). The integration has been carried out for $2n = 1{\cdot}5$ 3, 4, and 6. Fig. 1 shows y as a function

of t' for these particular cases. The sine curve ($2n=2$) has also been added for comparison (third curve from right). The higher the power of the restoring force, the more pointed the displacement curves become. $2n=\infty$ and 1 may be considered as the two limiting cases of this set of curves. The former yields the dotted line on the left, corresponding to triangular vibration curves,

$$y=t'.$$

The other limit is the dotted parabola,

$$y=1-\left(1-\frac{t'}{2}\right)^2.$$

The integrations have been carried out numerically from $y=0$ to, say, 0·98. For yet higher values of y the figures below the integral increase too rapidly, and a numerical calculation is not possible. Fortunately in this range an approximate integration is feasible by substituting

$$z=1-y.$$

Thus, for $z<<1$,

$$2nz=1-y^{2n}$$

and (for this range)

$$t'=\sqrt{\frac{2z}{n}},$$

t' corresponding to $z=0{\cdot}02$ has been added to the finite integral up to $y=0{\cdot}98$.

All these curves can naturally be expanded, if wanted, into Fourier series.

For our purpose the definite integrals t' giving $\frac{\pi}{2}b$, according to (7), were of special importance. The values $\frac{\pi}{2}b$ are plotted at the top part of fig. 2, the reciprocals of $2n$ being conveniently used as abscissæ. In joining the calculated points and extending the curve until $\frac{1}{2n}=0$, $\frac{\pi}{2}b=1$, a smooth line of very little curvature is obtained, thus allowing the value of b for any desired intermediate n to be read with fairly great accuracy. The lower curve in fig. 2 will be referred to in section 5.

4. With the aid of the function $b(n)$ the natural frequency can easily be found. From (4) the amplitude A is found as the value for x corresponding to $y=1$. Thus

$$A=\sqrt[2n]{\frac{nC}{a^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

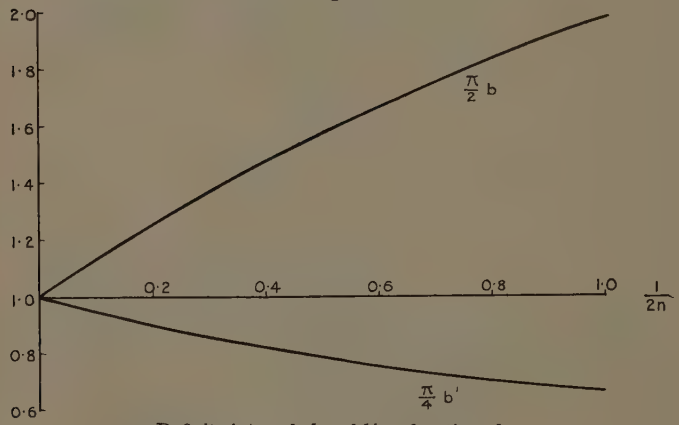
Substituting A instead of C in equation (9), we obtain

$$\omega=\frac{aA^{n-1}}{\sqrt{n}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and from (11)

$$\nu=\frac{aA^{n-1}}{2\pi b\sqrt{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Fig. 2.



Definite integrals b and b' as function of n.

For each n the numerical factor b can be read from fig. 2, and thus the natural frequency evaluated. It will depend on the amplitude increasing with it, as might be expected.

Substituting A by means of (12) in the relation for the maximum velocity

$$v_0=\sqrt{C},$$

we get

$$v_0=\frac{aA^n}{\sqrt{n}},$$

and from (13)

$$v_0 = \omega A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

the well-known relation for harmonic vibrations, which is seen, therefore, to be independent of n, if ω is defined by (8).

5. As already mentioned, an application of these relations should be given in asking for the probability function $\phi(\epsilon)$, if $\phi(\epsilon)d\epsilon$ denotes the probability for a non-linear molecular oscillator possessing an energy between ϵ and $\epsilon+d\epsilon$. Quite generally ϕ is given by

$$\phi(\epsilon)d\epsilon = Be^{-\epsilon/kT}d\Omega, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where B is a factor, k=Boltzmann's constant, T= absolute temperature, and $d\Omega$ a volume element of the phase space, corresponding to oscillators having the energy between ϵ and $\epsilon+d\epsilon$ *. The phase space has twice as many dimensions as the oscillator has degrees of freedom; in our case, therefore, two dimensions. One of them is the displacement x, the other the corresponding momentum mv.

The phase space in our case is really a phase plane, represented by the rectangular coordinates x and mv. Each oscillator of a given energy ϵ circulates along certain closed curves in that plane. The oscillator of energy $\epsilon+d\epsilon$ circulates along another curve close to the first, and $d\Omega$ is evidently the narrow closed strip limited by the two curves. It can be calculated by computing the surface S enclosed by the first curve and then taking the differential dS corresponding to $d\epsilon$.

Choosing x as abscissæ and mv as ordinates, we have for a quarter of an oscillation

$$\tfrac{1}{4}S = m\int_0^A v\,dx$$

or, considering (3) and (4),

$$\tfrac{1}{4}S = m\sqrt[2n]{\frac{nC^{n+1}}{a^2}}\int_0^1 \sqrt{1-y^{2n}}\,dy. \quad . \quad . \quad (17)$$

The surfaces in question are indicated in fig. 3 for the same values of n as in fig. 1. The third curve from the left is the circle, corresponding to the simple harmonic

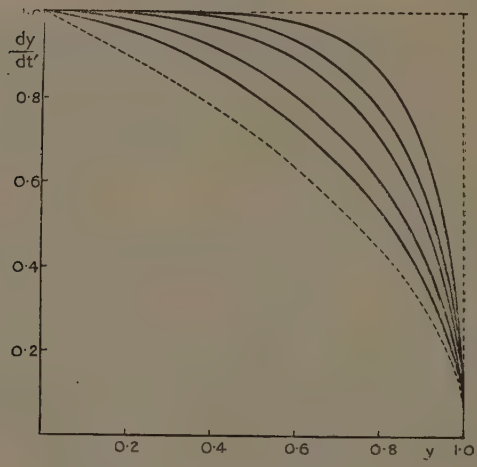
* See, for instance, M. Planck, ' Theory of Heat,' London, Macmillan, 1932, p. 255.

motion. The limits are the dotted quadrangle for $2n=\infty$ and the dotted parabola for $2n=1$. As to the numerical values of the definite integral, let us denote

$$\int_0^1 \sqrt{1-y^{2n}}\,dy=\frac{\pi}{4}b'. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

They have been evaluated like those in (7), plotted in fig. 2 (lower curve) and joined into a smooth line. For the simple harmonic case b' is again unity, like b.

Fig. 3.



Paths of oscillators in phase-plane for $2n$ (from left): 1, 1·5, 2, 3, 4, 6, ∞.

Substituting (12) into (17), we obtain

$$S=\frac{\pi b' am A^{n+1}}{\sqrt{n}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and, from (13),

$$S=\pi b' m\omega A^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which becomes an ellipse for the harmonic case. Since we are looking for the differential dS as a function of $d\epsilon$, S has to be expressed as a function of the energy ϵ. The

latter is given by the potential of the restoring force at the amplitude, or by $\frac{1}{2}mv_0^2$, *i. e.*,

$$\epsilon = \frac{f}{2n} A^{2n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

or, from (19),

$$S = \frac{\pi b' f}{a\sqrt{n}} \left(\frac{2n\epsilon}{f}\right)^{\frac{n+1}{2n}}, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

from which it follows that

$$dS = \text{const.}\ \epsilon^{\frac{1-n}{2n}} d\epsilon\ ; \quad . \quad . \quad . \quad . \quad (23)$$

dS being the appropriate differential phase-space element $d\Omega$ for this case, we get, by combining (16) and (23),

$$\phi . d\epsilon = B' e^{-\epsilon/kT} \epsilon^{(1-n)/2n} d\epsilon, \quad . \quad . \quad . \quad . \quad (24)$$

where B′ includes the constant from (23).

The value of B′ is obtained from the condition

$$\int_{\blacksquare} \phi\, d\epsilon = 1.$$

We have then

$$B' \int_0^\infty e^{-\epsilon/kT} \epsilon^{(1-n)/2n} d\epsilon = 1. \quad . \quad . \quad . \quad . \quad (25)$$

With the substitution

$$\frac{\epsilon}{kT} = u,$$

we obtain

$$B'(kT)^{\frac{1+n}{2n}} \int_0^\infty e^{-u} u^{\frac{1-n}{2n}} du = 1,$$

or *

$$\left(\frac{1-n}{2n}\right)! \, B'(kT)^{\frac{1+n}{2n}} = 1 \quad . \quad . \quad . \quad . \quad (26)$$

and

$$B' = \frac{1}{\left(\frac{1-n}{2n}\right)! \, (kT)^{\frac{1+n}{2n}}}. \quad . \quad . \quad . \quad . \quad (27)$$

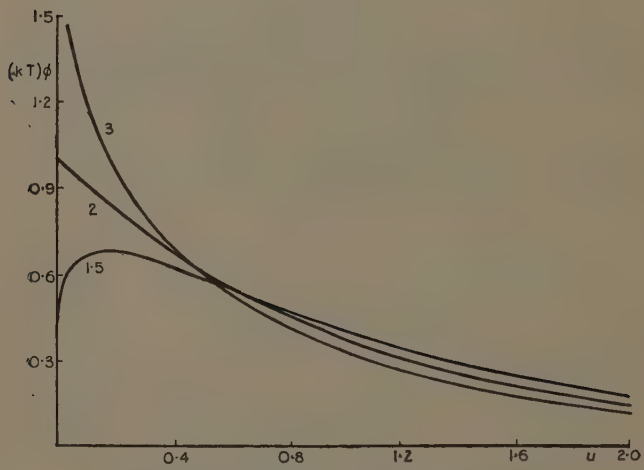
* E. Jahnke and F. Emde, ' Tables of Functions,' Teubner, Leipzig, 1933, p. 94.

Introducing again the argument u, we have finally

$$(kT).\phi = \frac{e^{-u}u^{\frac{1-n}{2n}}}{\left(\frac{1-n}{2n}\right)!} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Fig. 4 shows three probability curves for $2n=1{\cdot}5$, 2, and 3. The medium curve is the exponential, corresponding to the simple harmonic motion. For $n>1$

Fig. 4.



Energy distribution functions for $2n$: 3, 2, 1·5.

(increase of restoring force more than linear) the probability for small energies becomes very high, for $n<1$ (increase less than linear) very small. The curves cross in the neighbourhood of $u=0{\cdot}5$, and for still higher energies the behaviour is the opposite to that for small ones.

Summary.

The paper contains a discussion of free non-linear vibrations, the restoring force assumed to be proportional to any power of the displacement. Simple formulæ are derived, allowing immediate computation of the natural

frequency for any value of the exponent and for any amplitude.

A physical application of the results is given by evaluating the (classical) energy distribution function for a great number of non-linear molecular oscillators of the type concerned.

The author is indebted to Messrs. Metropolitan-Vickers Electrical Co. Ltd. for a grant enabling this investigation to be carried out.

XXVII. *A new Approach to the Hermite Polynomials.* *By* E. U. Condon *and* Robert Greenwood, *Palmer Physical Laboratory, Princeton University* *.

THE set of orthogonal functions defined by the equations

$$\psi_n(x)=\frac{1}{\sqrt{2^n n!}\sqrt{\pi}}\mathrm{H}_n(x)e^{-x^2/2} \quad n=0, 1, 2\ldots, \quad . \quad (1)$$

where the $\mathrm{H}_n(x)$ are the polynomials of Hermite usually defined by

$$\mathrm{H}_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \quad n=0, 1, 2\ldots, \quad . \quad . \quad (2)$$

play a great rôle in analysis. In this paper the theory of these functions is developed by means of an application of the symbolic operator methods which have lately found application in quantum mechanics †.

Consider two non-commuting operators, which we may denote by ξ and η, which obey the commutation law

$$\xi\eta-\eta\xi=-i. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Then we shall show that the orthonormal set of functions (1) are the proper functions of the quantity $\xi^2+\eta^2$ in the sense that

$$(\xi^2+\eta^2)\psi=\mathrm{A}\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

* Communicated by the Authors.

† See Dirac, 'Quantum Mechanics' (Oxford University Press, 1935), especially chapters 2, 3 and 4.

where A is an ordinary number. This symbolic equation receives various special interpretations which will exhibit properties of the Hermitian polynomials.

Using (3) we have

$$(\xi \pm i\eta)(\xi \mp i\eta) = \xi^2 + \eta^2 \mp 1, \quad . \quad . \quad . \quad . \quad (5)$$

and so (4) can be written

$$[(\xi \pm i\eta)(\xi \mp i\eta) \pm 1]\psi = \mathrm{A}\psi.$$

Transposing the term in 1ψ, and operating on both sides with $\xi \mp i\eta$, we get

$$(\xi \mp i\eta)(\xi \pm i\eta)[(\xi \mp i\eta)\psi] = (\mathrm{A} \mp 1)[(\xi \mp i\eta)\psi],$$

which in virtue of (5) becomes

$$(\xi^2 + \eta^2)[(\xi \mp i\eta)\psi] = (\mathrm{A} \mp 2)[(\xi \mp i\eta)\psi]. \quad . \quad . \quad (6)$$

From this we conclude that if ψ is a proper function of $\xi^2 + \eta^2$ in any explicit representation of the above symbolic work then $(\xi \mp i\eta)\psi$ are also proper functions belonging to the proper values $\mathrm{A} \mp 2$. Hence the proper values are equally spaced with the interval 2.

Inasmuch as the operators ξ and η defined by (3) may be taken to be Hermitian operators with real proper values, it follows that $\xi^2 + \eta^2$ is essentially positive, so there must be a least proper value of $\xi^2 + \eta^2$. Let this be denoted by A′, and denote its proper function by $\psi(\mathrm{A}')$. Then

$$(\xi - i\eta)\psi(\mathrm{A}') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for otherwise $(\xi - i\eta)\psi(\mathrm{A}')$ would be the proper function going with the proper value $\mathrm{A}' - 2$, which contradicts the hypothesis that A′ is the least proper value. Applying $(\xi + i\eta)$ to (7), and using (5), we have

$$(\xi^2 + \eta^2)\psi(\mathrm{A}') = \psi(\mathrm{A}'),$$

and therefore $\mathrm{A}' = 1$. Hence the proper values of $\xi^2 + \eta^2$ are given by

$$\mathrm{A} = 2n + 1. \quad n = 0, 1, 2 \ldots \quad . \quad . \quad . \quad . \quad (8)$$

We may conveniently label the proper functions by an integer n, as in the equation

$$(\xi^2 + \eta^2)\psi_n = (2n + 1)\psi_n. \quad n = 0, 1, 2 \ldots \quad . \quad . \quad . \quad (9)$$

Owing to the linearity of the equations we cannot conclude that $(\xi \mp i\eta)\psi_n$ is equal to $\psi_{n \mp 1}$, but merely that the former is equal to the latter to within a constant

factor. We now wish to determine that factor, so we write

$$\psi_{n+1}=C_n(\xi+i\eta)\psi_n.$$

The corresponding conjugate imaginary equation is

$$\bar{\psi}_{n+1}=\bar{C}_n\bar{\psi}_n(\xi-i\eta).$$

Hence symbolically

$$\begin{aligned}\bar{\psi}_{n+1}\psi_{n-1}&=\mid C_n\mid^2\bar{\psi}_n(\xi-i\eta)(\xi+i\eta)\psi_n\\&=\mid C_n\mid^2 2(n+1),\end{aligned}$$

if we assume that ψ_n is normalized so that $\bar{\psi}_n\psi_n=1$. Therefore C_n is determined to within a factor of magnitude unity. We may take $C_n=\mid C_n\mid$, and so finally one has

$$\psi_{n+1}=\frac{1}{\sqrt{2(n+1)}}(\xi+i\eta)\psi_n. \quad . \quad . \quad . \quad (10\,a)$$

From this it is easily seen that

$$\psi_{n-1}=\frac{1}{\sqrt{2n}}(\xi-i\eta)\psi_n. \quad . \quad . \quad . \quad (10\,b)$$

By iteration of (10 *a*) we readily find that

$$\psi_n=\frac{1}{\sqrt{2^n\,n\,!}}(\xi+i\eta)^n\psi_0. \quad . \quad . \quad . \quad (11)$$

Some recursion relations between the proper functions may be found by combination of the formulas (10) by adding and subtracting, thus:

$$\left.\begin{aligned}\xi\psi_n&=\sqrt{\tfrac{1}{2}(n+1)}\,\psi_{n+1}+\sqrt{\tfrac{1}{2}n}\,\psi_{n-1},\\ \eta\psi_n&=-i\sqrt{\tfrac{1}{2}(n+1)}\psi_{n+1}+i\sqrt{\tfrac{1}{2}n}\,\psi_{n-1}\end{aligned}\right\}. \quad . \quad (12)$$

Of course one can get expressions for $\xi^a\eta^b\xi^c\psi_n$ in terms of the ψ's for arbitrary positive integral or zero values of a, b, and c simply by repeated applications of (12). This work involves the same formalism, and hence the results will involve the same coefficients as when one expresses $\cos^a\theta$ or $\sin^a\theta$ in terms of sin and cos of multiples of the angle. For example

$$\xi^a=\left[\frac{(\xi+i\eta)+(\xi-i\eta)}{2}\right]^a.$$

This form can be expanded by the binomial theorem and the individual terms of the form $(\xi+i\eta)^k(\xi-i\eta)^m$ easily evaluated by successive applications of (10).

These results embrace all the simpler properties of the Hermitian polynomials, as we can see by giving an explicit representation to the foregoing symbolic work. One way to do this is to interpret the ψ_n as functions of x and to write

$$\xi \rightarrow \frac{1}{i}\frac{d}{dx} \qquad \eta \rightarrow x \qquad . \quad . \quad . \quad . \quad . \quad (13)$$

Then the equation (7) for determination of ψ_0 becomes

$$\left(\frac{1}{i}\frac{d}{dx} - ix\right)\psi_0(x) = 0,$$

whence

$$\psi_0(x) = \frac{1}{\sqrt[4]{\pi}} e^{-x^2/2} \qquad . \quad . \quad . \quad . \quad . \quad (14)$$

if we pass at once to the normalized function by introduction of the factor $\pi^{-\frac{1}{4}}$. In terms of this representation we have from (11)

$$\psi_n(x) = \frac{(-i)^n}{\sqrt{2^n n!}\sqrt{\pi}} \left(\frac{d}{dx} - x\right)^n e^{-x^2/2}. \quad . \quad . \quad (15)$$

This is in the form of a polynomial multiplied by $e^{-x^2/2}$, but a search of the literature failed to reveal that this form for the Hermitian polynomials has been given before. It is necessary to establish that the polynomials so defined really are the polynomials defined by (2). Let us consider more generally the polynomials defined by

$$\mathrm{K}_n(x) = (-1)^n e^{kx^2}\left(a\frac{d}{dx} + bx\right)^n e^{-kx^2},$$

so

$$\begin{aligned}\mathrm{K}_{n+1}(x) &= (-1)e^{kx^2}\left(a\frac{d}{dx} + bx\right)\mathrm{K}_n(x)e^{-kx^2} \\ &= (-1)[a\mathrm{K}_n'(x) + (b-2ak)x\mathrm{K}_n(x)].\end{aligned}$$

From well-known properties of the Hermite polynomials as defined by (2) we have a corresponding relation

$$\mathrm{H}_{n+1}(x) = (-1)[\mathrm{H}_n'(x) - 2x\mathrm{H}_n(x)].$$

Hence the connexion will agree with that for Hermitian polynomials if we take $a=1$ and $2k-b=2$. It is easy to verify that with this choice, using $n=0$ and $n=1$, the two sets of polynomials get started in the same way, so owing to agreement of the recursion relation they will

agree for all values of n. This argument establishes that the polynomials in (15) are identical with the polynomials defined in (2), and permits the writing of a more general formula :

$$\mathrm{H}_n(x)=(-1)^n e^{kx^2}\left[\frac{d}{dx}+(2k-2)x\right]^n e^{-kx^2}. \quad . \quad (16)$$

For $k=1$ this reduces to the ordinary formula (2) and for $k=1/2$ it reduces to the polynomial part of (15), which we originally set out to justify. An especially simple form is obtained by setting $k=0$, which gives a form containing no reference to the exponential function,

$$\mathrm{H}_n(x)=(-1)^n\left(\frac{d}{dx}-2x\right)^n . 1. \quad . \quad . \quad . \quad (17)$$

The identification of the polynomials in (15) with the ordinary Hermitian polynomials defined by (2) permits through (13) the establishment of the usual simple recursion relations for the polynomials by application of (12). Thus the symbolic method has provided an account of the recursion relations, a simple procedure for normalization of the orthogonal functions, and has suggested a generalization of (2) the usual equation of definition of the polynomials.

A more general realization of the commutation rule (3) than that used in (13) is

$$\xi=\left(\frac{c}{i}\frac{d}{dx}-sx\right), \quad \eta=\left(\frac{s}{i}\frac{d}{dx}+cx\right), \quad . \quad . \quad (18)$$

where $c^2+s^2=1$ (notation suggested by $\cos^2\theta+\sin^2\theta=1$), since the commutation rule (3) is invariant under an orthogonal substitution on ξ and η. With this representation of the operators (7) takes the form

$$(c-is)\left(\frac{d}{dx}+x\right)\psi_0(x)=0,$$

which leads to the same determination of $\psi_0(x)$ as before. Similarly (15) becomes

$$\psi_n(x)=\frac{(-i)^n e^{in\delta}}{\sqrt{2^n n!}\sqrt{\pi}}\left(\frac{d}{dx}-x\right)^n e^{-x^2/2}, \quad . \quad . \quad (19)$$

where $c+is=e^{i\delta}$. This is the same as (15) to within the trivial change of phase $e^{in\delta}$, so we see that the proper functions of $\xi^2+\eta^2$ retain the same analytical form no

matter which representation of the operators ξ and η is used within the one-parameter family defined by (18).

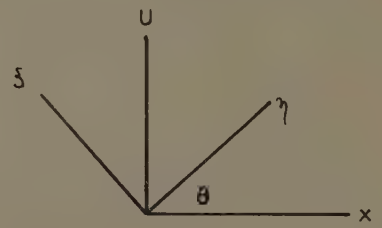
This result may be brought into close connexion with the generalization of the Fourier transform developed by Condon *. If in the group space there introduced we identify

$$U \rightarrow \frac{1}{i}\frac{d}{dx}, \quad X \rightarrow x,$$

then equations (18) amount to an orthogonal substitution on the operators

$$\xi = cU - sX, \quad \eta = sU + cX,$$

where $c = \cos\theta$ and $s = \sin\theta$. The elements of the group consist of rotations through an arbitrary angle in the group space. Now the operator $\xi^2+\eta^2 = U^2+X^2$ is invariant under the operations of the group. Hence we



may expect that the proper functions of this invariant operator will be invariant under the transformations of the group to within a constant factor of magnitude unity. Write

$$\phi_n(\eta) = \int \psi_n(x) K(x, \eta)\, dx \quad . \quad . \quad . \quad . \quad (20)$$

with

$$K(x, \eta) = \frac{e^{i\pi/4}}{\sqrt{2\pi s}} \exp\left\{ -\frac{ic}{2s}x^2 + \frac{ix\eta}{s} - \frac{ic}{2s}\eta^2 \right\},$$

this being the kernel for the functional transform which corresponds to rotation through angle θ. Then the theorem we have to establish is

$$\phi_n(\eta) = e^{in\delta}\psi_n(\eta). \quad . \quad . \quad . \quad . \quad . \quad (21)$$

* Condon, Proc. Nat. Acad. Sci. xxiii. p. 158 (1937). Since the appearance of that paper Prof. N. Wiener has kindly called our attention to the fact that the results contained in it were given by him in a paper in Journ. Math. and Phys. of Mass. Inst. Tech. viii. p. 70 (1929).

In the course of the proof the phases δ_n will be determined (eq. (24) below). By direct computation of the right side of (20) for the case $n=0$ we find the theorem is true for $n=0$ with

$$\delta_0=\tfrac{1}{2}\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Let us compute

$$\frac{\xi+i\eta}{\sqrt{2(n+1)}}\ \phi_n(\eta).$$

The operator acts only on the kernel, and gives

$$(\xi+i\eta)\phi_n(\eta)=\frac{1}{s}\left[\int x\psi_n(x)\mathrm{K}(x,\ \eta)dx-e^{-i\theta}\eta\phi_n(\eta)\right].$$

By the second of (12), with x in place of η,

$$x\psi_n(x)=-i\sqrt{\tfrac{1}{2}(n+1)}\psi_{n+1}(x)+i\sqrt{\tfrac{1}{2}n}\ \psi_{n+1}(x),$$

and assuming the validity of (21) for the value n,

$$\eta\phi_n(\eta)=e^{i\delta_n}[-i\sqrt{\tfrac{1}{2}(n+1)}\psi_{n+1}(\eta)+i\sqrt{\tfrac{1}{2}n}.\psi_{n-1}(\eta)].$$

Hence

$$\frac{\xi+i\eta}{\sqrt{2(n+1)}}\ \phi_n(\eta)=\frac{1}{s}\left\{-\frac{i}{2}\ \phi_{n+1}(\eta)+\frac{i}{2}\sqrt{\frac{n}{n+1}}\,e^{i\delta_{n-1}}\psi_{n-1}(\eta)\right.$$
$$\left.+\frac{i}{2}\,e^{-i\theta}e^{i\delta_n}\psi_{n+1}(\eta)-\frac{i}{2}\sqrt{\frac{n}{n+1}}\,e^{-i\theta}\,e^{i\delta_n}\psi_{n-1}(\eta)\right\}.$$

The coefficient of $\psi_{n+1}(\eta)$ in this expression vanishes if we choose

$$\delta_n-\delta_{n-1}=\theta,$$

and hence in view of (22) we take

$$\delta_n=(n+\tfrac{1}{2})\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

But also in view of the assumed validity of (21) for the value n we have from (10 a)

$$\frac{\xi+i\eta}{\sqrt{2(n+1)}}\ \phi_n(\eta)=e^{i\delta_n}\psi_{n+1}(\eta).$$

So (23) reduces to an equation for the determination of $\phi_{n+1}(\eta)$ and gives

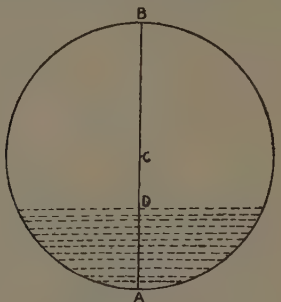
$$\phi_{n+1}(\eta)=e^{i\delta_{n+1}}\psi_{n+1}(\eta),$$

where δ_{n+1} is defined in accord with (24). Hence the theorem is proved by induction, since its validity for $n=0$ was established by direct computation.

XXVIII. *Waves in a Circular Canal.*

By B. R. Seth, *M.A., D.Sc.**

THE problem of the transverse oscillations of water contained in a canal of circular section has not yet been solved. When the free surface is at the level of the axis both Lamb † and Rayleigh ‡ have obtained an approximate value for the frequency of the gravest mode by using Rayleigh's method § for cases in which the normal types cannot be accurately determined; but both of them have failed to notice that the constrained type assumed by them is unstable, and hence the result obtained is of not much consequence. We shall show that for a circular canal no exact stable type can be



BD=h Ac=a

obtained, but, if the depth of water in the canal be small compared with its diameter, an approximate determination of the frequency can be effected.

Let us take the origin at the centre of the section, and the axis of y vertically upwards.

The value of ϕ for the constrained type used by Lamb is given by

$$\phi+i\psi=\frac{i\dot{\theta}a^2}{\pi}\left[\frac{1}{2}\pi\frac{z^2}{a^2}-i\left(\frac{z}{a}+\frac{a}{z}\right)\right.$$
$$\left.+\frac{1}{2}i\left(\frac{z^2}{a^2}+\frac{a^2}{z^2}-2\right)\log\frac{1+z/a}{1-z/a}\right], \quad . \quad (1)$$

* Communicated by the Author.
† ' Hydrodynamics,' p. 420, 5th ed. (1930).
‡ Phil. Mag. (5), xlvii. p. 566 (1899) [Papers, iv. p. 407].
§ *Cf.* ' Theory of Sound,' i. pp. 109–111, 2nd ed. (1894).

where θ is the small angle which the free surface (assumed to remain plane) makes with the horizontal, and a is the radius of the section. This ϕ satisfies the condition

$$\frac{\partial\phi}{\partial n}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

over the fixed boundary $r=a$, but does not satisfy the condition

$$\sigma^2\phi=g\frac{\partial\phi}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

at the free surface, which is $y=0$. If the type given by (1) is to be stable (3) should give only real values for σ when $x^2\leqslant a^2$. Substituting the value of ϕ from (1) in (3), and putting $y=0$, we get

$$\frac{\sigma^2a^2}{\pi}\left[\frac{1}{2}\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}-2\right)\log\frac{1+x/a}{1-x/a}-\left(\frac{a}{x}+\frac{x}{a}\right)\right]=gx, \quad (4)$$

which gives

$$\sigma^2=-\frac{\pi g}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.1)$$

when $x^2=a^2$, and

$$\sigma^2\left(\frac{8}{3}-\frac{8}{15}\frac{x^2}{a^2}\cdots\right)=-\frac{\pi g}{a}, \quad . \quad . \quad . \quad (5.2)$$

when $x^2<a^2$.

Thus σ is always imaginary corresponding to this constrained type, and hence the approximate normal type deduced from (1) cannot be stable, Moreover, since (x/a) is not a small quantity for all points on the free surface terms of the second and higher degrees in (x/a) cannot be neglected in (5). The solution given by (1) therefore cannot be even used to determine the degree of instability.

To prove that no exact stable type can be obtained for this section we shift the origin to the highest point B (see figure) and take the axis of y vertically downwards. Any function which is to represent the solution must be finite and continuous, together with its first partial derivatives, at least for all points of the section excepting the highest point. We can therefore assume either

$$\phi+i\psi=\Sigma K_n\sin k_n\left(\frac{2a}{z}+i\right), \quad . \quad . \quad . \quad (6.1)$$

or

$$\phi+i\psi=C+\Sigma L_n \cos k_n\left(\frac{2a}{z}+i\right), \quad . \quad . \quad (6.2)$$

both of which satisfy the Laplace's equation, and make the normal velocity zero over the boundary $r=2a\sin\theta$, C, K's, L's, and k's being all real constants. The asymmetrical modes are given by (6.1) and the symmetrical by (6.2).

If we take (6.1) we see that ϕ can be written as

$$\phi=\Sigma A_n \frac{\sin(n\theta+\frac{1}{2}p\pi)}{r^n}, \quad . \quad . \quad . \quad . \quad . \quad (7.1)$$

where p is 1 or 0 according as n is odd or even, A's being all real constants, as yet undetermined. Rewriting (7.1), we have

$$\phi=x\left[\frac{A_1}{r^2}+\frac{1}{r^4}(A_3+2A_2y)+\frac{1}{r^6}(A_5+4A_4y-4A_3y^2)\right.$$
$$\left.+\frac{1}{r^8}(A_7+6A_6y-12A_5y^2-8A_4y^3)+\ldots\right]. \quad . \quad (7.2)$$

The condition at the free surface, $y=h$, which is now of the form

$$\sigma^2\phi=-g\frac{\partial\phi}{\partial y},$$

gives $A_1=0$, and

$$2A_2(lh+1)+lA_3=0, \quad . \quad . \quad . \quad . \quad (8.1)$$
$$8A_2h^2+4A_3h(lh+3)-4A_4(lh+1)-lA_5=0, \quad . \quad (8.2)$$
$$24A_3h^3-8A_4h^2(lh+6)-6A_5h(2lh+5)+6A_6(lh+1)+lA_7=0,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.3)$$

where $l=\sigma^2/g$.

The elimination of A's from (8) gives an infinite determinant to determine σ. If we proceed to evaluate σ by successive approximations we get equations of the following type :

$$\left.\begin{aligned} lh+1&=0,\\ l^2h^2+3lh+3&=0,\\ l^3h^3+6l^2h^2+15lh+15&=0,\\ .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .&\end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

Thus at whatever stage we may stop no positive value of lh can be obtained, and hence no asymmetrical stable type is possible.

On using (6.2) we find

$$\phi = A_0 + \Sigma A^n \frac{\sin(n\theta + \frac{1}{2}p\pi)}{r^n}$$

$$= A_0 + \frac{1}{r^2}(A_2 + A_1 y) + \frac{1}{r^4}(A_4 + 3A_3 y - 2A_2 y^2)$$

$$+ \frac{1}{r^6}(A_6 + 5A_5 y - 8A_4 y^2 - 4A_3 y^3)$$

$$+ \frac{1}{r^8}(A_8 + 7A_7 y - 18A_6 y^2 - 20A_5 y^3 + 8A_4 y^4)$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \qquad (10)$$

p being now 0 or 1 according as n is odd or even.

Proceeding as above we get the same equations as given in (9) to determine σ. There exists therefore no symmetrical stable type as well.

If we put equal to zero all the A's in (7) excepting A_1, we get

$$\phi = \frac{A_1 x}{r^2}, \quad \psi = -A_1\left(\frac{y}{r^2} - \frac{1}{2a}\right) . \quad . \quad . \quad . \qquad (11)$$

Obviously this value of ψ vanishes over the circular boundary $r = 2a \sin\theta$. The condition at the free surface gives a real value of σ for all values of x, but is approximately satisfied only if the depth of water in the canal is small compared with its diameter. The constrained type given by (11) is therefore stable. On using Rayleigh's method we find

$$\sigma^2 = \frac{g}{2h} \cdot \frac{a^2(2\beta + \sin 2\beta) - h^2 \tan\beta}{a^2(2\beta - \sin 2\beta)} , \quad . \quad . \qquad (12)$$

where 2β is the angle which the undisturbed free surface subtends at B.

If we assume that (12) can be used even when $h = a$ we get

$$\sigma^2 = \frac{g}{a} \cdot \frac{\pi}{2(\pi - 2)} = (1{\cdot}376)\frac{g}{a},$$

whence $\sigma = 1{\cdot}173\ (g/a)^{\frac{1}{2}}$.

It is interesting to note that this value of σ does not differ very much either from Lamb's, which is

$$\sigma = 1{\cdot}169\left(\frac{g}{a}\right)^{\frac{1}{2}},$$

or from Rayleigh's, which is

$$\sigma = (1{\cdot}1644)\left(\frac{g}{a}\right)^{\frac{1}{2}}.$$

But, as we have already pointed out, (12) cannot be used for depths which are not small compared with the diameter.

The value of σ in a few cases is given in the following table :—

H=depth of water in the canal.

No.	H/2a.	$\sigma(a/g)^{\frac{1}{2}}$.
1	0·0170	1·004
2	0·0670	1·014
3	0·1464	1·033
4	0·2500	1·063

For very small values of H the formula

$$\sigma = \left(\frac{g}{a}\right)^{\frac{1}{2}}\left[1 + \frac{1}{5}\cdot\frac{H}{2a}\right]$$

can be used. Thus the frequency decreases with the depth, and approaches the limiting value $(g/a)^{\frac{1}{2}}$ as $H \to 0$. In such cases the water in the canal oscillates like a long thin bar executing small oscillations about the lowest generator in a circular cylinder.

A similar result holds good for a hyperbolic section. We know that for a rectangular hyperbola and for one in which the eccentricity is 2 the value of σ is given by *

$$\sigma = \left(\frac{g}{h}\right)^{\frac{1}{2}},$$

h being the height of the free surface above the centre of hyperbolas. σ now increases as the depth decreases, and tends to the limiting value $(g/a)^{\frac{1}{2}}$, a being the semi-major axis.

* *Cf.* Seth, Phil. Mag. (7) xxiii. pp. 106–114 (1937).

But such is not the case with rectilinear sections. We know that for many sections bounded by two straight lines equally inclined to the vertical σ for the gravest mode is given by *

$$\sigma = \left(\frac{g}{h}\right)^{\frac{1}{2}},$$

h being the depth of water in the canal. σ again increases as h decreases, but approaches infinity and not any finite value as $h \to 0$.

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XXIX. *A Note on the Vibrations of a Rotating Rope.* *By* A. C. Stevenson, *M.Sc., Lecturer in Applied Mathematics, University College, London* † ‡.

THE object of this note is to remedy a slight blemish which occurs in a number of books of introduction to the methods of theoretical physics. The problem of the vibrations of a rotating rope is there considered, not indeed from any vital interest in the concrete problem, but as a useful mathematical example employing normal functions of a simple character, serving to illustrate the mathematical method used in a variety of admittedly more important problems in mathematical physics. Unfortunately, the formulation of the problem has not usually received as much attention as the solution, which suffers as a result by being at best incomplete, at worst by being both incomplete and inaccurate.

We consider a uniform inextensible string of length l and mass m per unit length, whirled at a great rate ω about a vertical axis through one end O, so that it stands out straight and horizontal in steady motion with ω constant, gravity being neglected in comparison with the mass-accelerations when ω is very large. The problem is to determine the normal functions and the allowable frequencies for the small transverse vibrations

* *Cf.* Seth, Phil. Mag. (7) xxiii. pp. 106–114 (1937).

† Communicated by the Author.

‡ The author calls attention to vagueness and error in the formulation of this problem, and gives corrected results.

about the steady state. If the string is along Ox in the steady motion, Oz being the axis of rotation, the transverse displacements of a point P, of coordinates (x, 0, 0) in the steady state, will have a horizontal component y, and a vertical component z. It does not appear to have been realized that the differential equations for y and z are not the same, as they are, for example, for the transverse displacements of a string tightly stretched between two points.

If $\mathbf{r}$ is the position vector of an element δs of the string at P, $\mathbf{u}$ is the unit vector along the tangent to the string at P, and T is the tension at P, the equation of motion of the element is

$$md^2\mathbf{r}/dt^2=\lim \delta\{\mathrm{T}\mathbf{u}\}/\delta s=\partial\{\mathrm{T}\mathbf{u}\}/\partial s. \quad . \quad . \quad . \quad (1)$$

If unit vectors along the rotating axes O (x, y, z) are $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively,

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k},$$

and

$$\mathbf{u}=\lim \delta\mathbf{r}/\delta s=\frac{\partial x}{\partial s}\mathbf{i}+\frac{\partial y}{\partial s}\mathbf{j}+\frac{\partial z}{\partial s}\mathrm{k}, \quad . \quad . \quad . \quad (2)$$

$\delta\mathbf{r}$ being an increment of the position vector (at a given instant). Also we have Coriolis's theorem

$$d\mathbf{v}/dt=\dot{\mathbf{v}}+[\omega\mathbf{k} \,.\, \mathbf{v}], \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\dot{\mathbf{v}}$ is the apparent rate of change of the vector $\mathbf{v}$ in the frame O(x, y, z) rotating with angular velocity $\omega\mathbf{k}$. A double application of this theorem to the position vector $\mathbf{r}$ gives $d^2\mathbf{r}/dt^2$, and this combined with equation (2) gives us, from (1), the vector equation whose x, y, z resolutes are respectively

$$m(\ddot{x}-2\omega\dot{y}-x\omega^2)=\frac{\partial}{\partial s}\left(\mathrm{T}\frac{\partial x}{\partial s}\right), \quad . \quad . \quad . \quad . \quad (4)$$

$$m(\ddot{y}+2\omega\dot{x}-y\omega^2)=\frac{\partial}{\partial s}\left(\mathrm{T}\frac{\partial y}{\partial s}\right), \quad . \quad . \quad . \quad . \quad (5)$$

$$m\ddot{z}=\frac{\partial}{\partial s}\left(\mathrm{T}\frac{\partial z}{\partial s}\right). \quad . \quad . \quad . \quad . \quad (6)$$

If we assume the departures y, z from the steady state to be very small, $\partial x/\partial s$ is approximately unity, and we may replace s throughout by x, noting that since the string

is inextensible, to the same degree of approximation $\dot{x}$ and $\ddot{x}$ vanish. Again, the term $-2\dot{y}\omega$ in equation (4) is certainly very small compared with $x\omega^2$, and the equations of motion may be written

$$-mx\omega^2 = \frac{\partial T}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$m(\ddot{y} - y\omega^2) = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$m\ddot{z} = \frac{\partial}{\partial x}\left(T\frac{\partial z}{\partial x}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

so that, integrating (4), and noting that the tension must be zero at the free end,

$$T = \tfrac{1}{2}m\omega^2(l^2 - x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

It is already clear that the equations of vibration (8) and (9) in y and z are not quite the same differential equation.

If we assume a normal mode of vibration

$$z = \phi(\xi)\{A \cos pt + B \sin pt\},$$

where

$$\xi = x/l,$$

and substitute from this and (10) into (9), we find the differential equation for the normal function ϕ is

$$\frac{d}{d\xi}\left\{(1-\xi^2)\frac{d\phi}{d\xi}\right\} + \lambda\phi = 0, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where

$$\lambda = \lambda_1 = 2p^2/\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If we assume a normal mode of vibration

$$y = \phi(\xi)\{A \cos pt + B \sin pt\},$$

we are again led to equation (11), where, however,

$$\lambda = \lambda_2 = 2(1 + p^2/\omega^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

To solve (11) for ϕ, we assume

$$\phi(\xi) = \sum_{r=1}^{\infty} a_r \xi^{t+r}.$$

Substitution of this into (11) leads to the relations

$$a_1 t(t+1)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$a_2(t+1)(t+2)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$a_{r+2}(t+r+2)(t+r+1)=a_r\{(t+r)(t+r+1)-\lambda\}. \quad (16)$$

Now $a_1 \neq 0$, since it is the first term of our series ; if from (14) we take $t=-1$, $\phi(0)=a_1$, which shows that we must reject this solution, since y, z, and therefore our normal functions, must vanish with x. Hence (14) leads to $t=0$, (15) then gives $a_2=0$, and (16) shows that the even coefficients a_{2r} all vanish.

Hence $\phi(\xi)$ can be written

$$\phi(\xi)=\sum_{n=1}^{\infty} u_n \xi^{2n-1},$$

where

$$\frac{u_{n+1}}{u_n}=\frac{2n(2n-1)-\lambda}{2n(2n+1)}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now consider the value of $\phi(\xi)$ at the free end of the rope, *i. e.*, $\phi(1)=\sum_{n=1}^{\infty} u_n$, and examine the convergence. Inspection shows that, for a given λ, after a certain value of n, the terms of the series will be one-signed. Use Gauss's test *, namely, that if

$$u_{n+1}/u_n=1-\mu/n+0(1/n^2),$$

the series $\sum_{n=1}^{\infty} u_n$ converges if $\mu<1$, and diverges if $\mu \leqslant 1$. From equation (17) we find $\mu=1$ in our case, and the series is therefore divergent. Clearly the infinite series is an unsuitable solution, and to be satisfactory the series must terminate. It is this fact which gives us the allowable frequencies of vibration. From (17) the termination of the series necessitates

$$\lambda=2n(2n-1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and, n being an integer, the solution for ϕ is recognizable now, either from the differential equation or the polynomial solutions discovered, as the Legendre polynomial of odd order, *i. e.*,

$$\phi_n(\xi)=\mathrm{P}_{2n-1}(x/l). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

* See, for example, E. G. Phillips, ' Analysis ' (Cambridge University Press), p. 130.

The normal functions for the horizontal and vertical vibration are accordingly the same, but the allowable frequencies $2\pi/p_n$, ($n=1, 2, 3, \ldots \infty$), are different, for we have from (12) and (18)

$$p_n^2/\omega^2 = n(2n-1) \quad . \quad . \quad . \quad . \quad . \quad (20)$$

for the allowable frequencies of the vibration components in the plane $y=0$, and from (13) and (18) we find

$$p_n^2/\omega^2 = (n-1)(2n+l). \quad . \quad . \quad . \quad . \quad (21)$$

for the allowable frequencies of the vibration components in the plane $z=0$, *and these are quite different sets of values.*

In Webster's ' Partial Differential Equations of Mathematical Physics ' (second corrected edition, 1933), p. 137, the direction of the transverse displacement is nowhere specified, and the differential equation given for the vibrations of the rotating chain should be stated to be that for the vibrations in a vertical plane.

In Slater and Frank's ' Introduction to Theoretical Physics ' (first edition, 1933), p. 158, ex. 2, the same criticism holds good.

In Morse's ' Vibration and Sound ' (first edition, 1936), p. 82, a much fuller treatment is given, but there is some confusion. In obtaining his equation (11.9) corresponding in form to our equation (9), he refers to the " net vertical force on an element of string ". At this stage the student would therefore consider his unspecified displacement y to be in the vertical plane. Later, however, from the remark " we shall neglect gravity, so that the string's motion will be in a plane " (p. 82), and from the form of his integral expression for the total angular momentum on p. 86, it finally becomes very clear that the plane of motion envisaged is the horizontal plane, and Morse's equation (11.9) is accordingly incorrect and should be replaced by our equation (8). Incidentally it is as well to realize that the axis of rotation need not be vertical, and that the terms " vertical plane " and " horizontal plane " have been used throughout this note to avoid the circumlocutions " axial plane containing the equilibrium line " and " plane perpendicular to the axis of rotation containing the equilibrium line." Following out the consequences of Morse's incorrect equation we find that the frequencies listed in his equations

(11.11) corresponding to our equation (20), should be replaced by our equation (21). From the fact that the normal functions are unaltered by the revision of his error, these are the only essential corrections to the five pages which Morse devotes to this problem, with the exception of his concluding paragraphs. the subject matter of which we revise below.

The total angular momentum of the rotating string about the vertical Oz is $\int_0^l m(\dot{y}+x\omega)x\,dx$, and the general solution for y leads to

$$\dot{y}=\sum_{n=1}^{\infty} p_n\phi_n(\xi)\{-\mathrm{A}_n\sin p_n t+\mathrm{B}_n\cos p_n t\}, \quad . \quad (22)$$

where A_n and B_n are small arbitrary constants depending upon the initial circumstances of vibration, and p_n is given by (21). From the orthogonal and normal properties of the functions ϕ_n, namely

$$\int_0^1 \phi_n(\xi)\phi_m(\xi)d\xi=0, \quad \text{for } n\neq m, \quad . \quad . \quad . \quad (23)$$
$$=1/(4n-1), \text{ for } n=m,$$

and the fact that $x=l\phi_1(\xi)$, we verify easily that the total angular momentum about the axis of rotation is constant, as it must be if the rope is whirling freely.

In this connexion Morse found it necessary to regard the fundamental mode of vibration, ($n=1$), as a forbidden mode, and had to put A_1 and B_1 both zero to render the angular momentum constant. Since $p_1=0$, from equation (21) the fundamental mode is $y=\mathrm{A}_1x/l$, and the term arising from this mode in the angular momentum integral is multiplied by p_1, and no such discrimination against the mode is necessary. It is not a vibration properly speaking, and expresses a disturbance which merely advances the equilibrium line of the steady motion through the small constant angle (A_1/l), and Morse's statement on p. 86 that "an attempt to produce the fundamental vibration will serve only to produce a change in the angular velocity ω of the equilibrium line' is not valid.

The present writer has inserted the material of equations (14) to (17) as a slight improvement on Morse's treatment; he assumes $t=0$ straight away in our expansion

$$\phi(\xi)=\sum_{r=1}^{\infty} a_r\xi^{t+r},$$

and the divergence of the infinite series is not exhibited to the student by a simple test, it is merely stated that the series is found *on computation* to become infinite for $\xi=1$. But these are minor criticisms compared with the main one, viz., that when he states on p. 82 that "we can neglect the uniform motion of the equilibrium line and deal only with the string's displacement from this line," he is guilty, in his interpretation of this, of forgetting that accelerations *relative to rotating axes* depend in general upon the rotation of the axes.

In conclusion the present writer hastens to add that the error discussed does not detract seriously from the usefulness of the excellently conceived text-book in which it occurs.

XXX. *Some Properties of Bessel Function Zeros derived from the Wave Functions for a Particle in an Enclosure.* *By* R. P. Bell *.

THE wave equation for a particle of mass m and energy W is

$$\Delta^2\psi+\lambda\psi=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\lambda=\dfrac{8\pi^2mW}{h^2}$. For a particle in an enclosure, λ is restricted to certain proper values by the boundary conditions that $\psi=0$, the detailed distribution of the proper values depending on the shape of the enclosure. It is, however, clear from physical considerations that for large values of λ this distribution should depend only on the volume and not on the shape of the enclosure, and it has been shown rigidly by Courant † and others that if $N(\lambda)$ is the number of proper values less than λ, then

$$\operatorname*{Lim}_{\lambda\to\infty}\frac{6\pi^2N(\lambda)}{\lambda^{3/2}}=1, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

independent of the form of the volume V.

If the enclosure is a rectangular box with sides of lengths a, b, and c, the boundary conditions can be applied

* Communicated by the Author.

† *Math. Z.* vii. p. 1 (1920).

directly to the solution of (1) in Cartesian coordinates, giving for the proper values the well-known expression

$$\lambda=\pi^2\left(\frac{n_x^2}{a^2}+\frac{n_y^2}{b^2}+\frac{n_z^2}{c^2}\right), \quad . \quad . \quad . \quad . \quad (3)$$

where n_x, n_y, and n_z are integers. Evaluation of $N(\lambda)$ for this case merely leads to direct confirmation of (2).

In the case of a spherical or cylindrical enclosure however, equation (2) can be applied to obtain some new properties of the Bessel functions which occur in the solution of the wave equation in the appropriate co-ordinate systems. Thus in spherical polar coordinates the wave-equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\lambda\psi=0, \quad . \quad . \quad . \quad (4)$$

with the solution

$$\psi=Ae^{im\phi}P_l^{|m|}(\cos\theta)\frac{1}{\sqrt{r}}J_{l+\frac{1}{2}}(\sqrt{\lambda}r), \quad . \quad . \quad (5)$$

where $l=0, 1, 2, \ldots$, and $m=0, \pm1, \pm2, \ldots \pm l$.

For a spherical enclosure of radius a, the boundary conditions give

$$\lambda=\frac{j_{l+\frac{1}{2},k}^2}{a^2} \qquad (k=1, 2, \ldots), \quad . \quad . \quad . \quad . \quad (6)$$

where $j_{l+\frac{1}{2},k}$ is the kth zero of $J_{l+\frac{1}{2}}(x)$. Each value of λ is $(2l+1)$-fold degenerate, corresponding to different values of m. If $N_{l+\frac{1}{2}}(y)$ is the number of zeros of $J_{l+\frac{1}{2}}(x)$ less than y, substitution of (6) in (2) gives directly the simple relation

$$\underline{\operatorname{Lim}_{y\to\infty}\frac{1}{y^3}\sum_{l=0}^{\infty}(2l+1)N_{l+\frac{1}{2}}(y)=\frac{2}{9\pi}}. \quad . \quad . \quad . \quad (7)$$

It may be noted that, in common with the other limiting values derived in this paper, (7) cannot be deduced from the usual expression for large zeros of $J_n(x)$, *i. e.*,

$$j_{n,k}\sim(k+\tfrac{3}{4}+\tfrac{1}{2}n)\pi, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

since this is only applicable when $k>>n$. It may, however, be verified by using the asymptotic expressions

obtained by Nicholson* for $J_n(y)$ when both n and y are large. $J_n(y)$ possesses zeros only when $y > n$, for which case Nicholson gives the expression

$$J_n(y) = \frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{2}}(y^2-n^2)^{\frac{1}{4}}} \sin\left\{(y^2-n^2)^{\frac{1}{2}} + n \sin^{-1}\frac{n}{y} - \frac{\pi}{4}(2n-1)\right\}.$$

The number of zeros less than y is hence

$$N_n(y) = \frac{1}{\pi}\left\{(y^2-n^2)^{\frac{1}{2}} + n \sin^{-1}\frac{n}{y} - \tfrac{1}{2}n\pi\right\},$$

and the sum in (7) can be replaced by the integral

$$\int_0^y \frac{2n}{\pi}\left\{(y^2-n^2)^{\frac{1}{2}} + n \sin^{-1}\frac{n}{y} - \tfrac{1}{2}n\pi\right\} dn = \frac{2y^3}{9\pi},$$

in agreement with the result already given.

Further relations can be obtained by considering the *partition function* Q for particles in an enclosure. This is defined by the equation

$$Q = \sum_i e^{-w_i/kT} = \sum_i e^{-\sigma\lambda_i}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $\sigma = \dfrac{h^2}{8\pi^2 mkT}$, and the summation is extended over all permissible independent wave functions. The partition function for translational motion is normally obtained by considering the case of a rectangular box, substitution of (3) in (9) giving

$$\lim_{\sigma \to 0} \frac{8\pi^{3/2} Q \sigma^{3/2}}{abc} = 1. \quad . \quad . \quad . \quad . \quad (10)$$

In practice this limiting value is approached with great accuracy at ordinary temperatures for all particles except the electron. The physical significance of the partition function makes it clear that its limiting value must depend only on the volume and not on the shape of the enclosure, *i. e.*, we must have

$$\lim_{\sigma \to 0} \frac{8\pi^{3/2} Q \sigma^{3/2}}{V} = 1. \quad . \quad . \quad . \quad . \quad (11)$$

This result has not been derived directly, but it can easily be related to the independence theorem (2). In the limiting case the sum in (9) can be replaced by the integral

$$Q = \int_0^\infty \frac{dN}{d\lambda} e^{-\sigma\lambda}\, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

* Phil. Mag. (6) xix. p. 228 (1910).

which, on substituting from (2), gives

$$Q = \frac{V}{4\pi^2}\int_0^\infty \lambda^{\frac{1}{2}} e^{-\sigma\lambda}\, d\lambda = \frac{V}{8\pi^{3/2}\sigma^{3/2}}, \quad . \quad . \quad . \quad (13)$$

in agreement with (11).

For a spherical enclosure the partition function is given by (6) and (9) as

$$Q = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}(2l+1)e^{-\sigma j_{l+\frac{1}{2},k}^2/a^2}. \quad . \quad . \quad . \quad (14)$$

Comparing this with (11), we find

$$\lim_{\sigma\to 0}\frac{6\pi^{\frac{1}{2}}\sigma^{3/2}}{a^3}\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}(2l+1)e^{-\sigma j_{l+\frac{1}{2},k}^2/a^2} = 1 \quad . \quad (15)$$

or

$$\lim_{t\to 0}\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}(2l+1)t^{3/2}e^{-j_{l+\frac{1}{2},k}^2 t} = \frac{\sqrt{\pi}}{6}. \quad . \quad . \quad (16)$$

The same procedure can be applied to the case of a cylindrical enclosure of radius a and length b. The wave equation in cylindrical coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + \lambda\psi = 0, \quad . \quad . \quad (17)$$

with the solution

$$\psi = Ae^{im\phi}\sin(\sqrt{\lambda_2}z)J_{|m|}(\sqrt{\lambda'}\rho), \quad . \quad . \quad . \quad (18)$$

where $\lambda_2+\lambda' = \lambda$ and $m = 0, \pm 1, \pm 2$ etc. The boundary conditions give

$$\left.\begin{aligned} \lambda_2 &= \frac{n^2\pi^2}{b^2} \qquad (n = 1, 2, \ldots), \\ \lambda' &= \frac{j_{|m|,k}^2}{a^2} \qquad (k = 1, 2, \ldots), \end{aligned}\right\} \quad (19)$$

with the same notation as before. In this case there is no degeneracy, and the partition function is given by

$$Q = \sum_{n=1}^{\infty} e^{-n^2\pi^2\sigma/b^2}\sum_{k=1}^{\infty}\sum_{m=-\infty}^{+\infty} e^{-\sigma j_{|m|,k}^2/a^2}, \quad . \quad (20)$$

whence for $\sigma\to 0$

$$Q = \frac{b}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\sum_{k=1}^{\infty}\sum_{m=0}^{\infty} e^{-\sigma j_{m,k}^2/a^2}, \quad . \quad . \quad . \quad (21)$$

since $j_{m,k}=j_{-m,k}$. In this case $V=\pi a^2 b$ and comparison with (11) gives

$$\lim_{t\to 0} \sum_{k=1}^{\infty} \sum_{m=0} t e^{-j^2_{m,k} t} = \tfrac{1}{8}. \quad . \quad . \quad . \quad . \quad (22)$$

It is clear that further relations can be obtained by replacing Q by other sums not necessarily having any physical significance. As an example we may take the four expressions defined by equations (23)–(26),

$$R_n = \sum_i \lambda_i^n e^{-\sigma\lambda_i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$S_n = \sum_i \lambda_i^{-\frac{1}{2}} e^{-\sigma\lambda_i^n}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$T_n = \sum_i \lambda_i^{2n-\frac{1}{2}} e^{-\sigma\lambda_i^2}, \quad . \quad . \quad . \quad (25)$$

$$U_n = \sum_i \lambda_i^{2n+\frac{1}{2}} e^{-\sigma\lambda_i^2}, \quad . \quad . \quad . \quad . \quad (26)$$

where n is arbitrary in R_n and S_n and a positive integer (or zero) in T_n and U_n. Limiting expressions for these sums can be obtained by replacing them by the corresponding integrals and using (2): we then obtain, independent of the shape of the boundary,

$$\lim_{\sigma\to 0} \frac{4\pi^2\sigma^{n+\frac{3}{2}} R_n}{\Pi(n+\frac{1}{2})V} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\lim_{\sigma\to 0} \frac{4\pi^2\sigma^{\frac{1}{n}} S_n}{\Pi\left(\frac{1}{n}\right) V} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\lim_{\sigma\to 0} \frac{2^{n+4}\pi^{3/2}(n-1)!\,\sigma^{n+\frac{1}{2}} T_n}{(2n-1)!\,V} = 1, \quad . \quad . \quad (29)$$

$$\lim_{\sigma\to 0} \frac{8\pi^2\sigma^{n+1} U_n}{n!V} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

In the case of a spherical enclosure these sums can be expressed in terms of the Bessel function zeros by using (16). On comparing these expressions with equations (27)–(30) we obtain the following relations :—

$$\lim_{t\to 0} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} (2l+1) t^{n+\frac{1}{2}} j^{2n}_{l+\frac{1}{2},k} e^{-j^2_{l+\frac{1}{2},k} t} = \frac{\Pi(n+\frac{1}{2})}{3\pi}, \quad . \quad . \quad (31)$$

$$\lim_{t\to 0} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{(2l+1)t}{j_{l+\frac{1}{2},k}} e^{-j^2_{l+\frac{1}{2},k}t} = \frac{\Pi\left(\frac{1}{n}\right)}{3\pi}, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$\lim_{t\to 0} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} (2l+1)t^{n+\frac{1}{2}} j^{4n-1}_{l+\frac{1}{2},k} e^{-j^4_{l+\frac{1}{2},k}t} = \frac{(2n-1)!}{3.2^{n+2}(n-1)!\sqrt{\pi}}, \quad . \quad . \quad . \quad (33)$$

$$\lim_{t\to 0} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} (2l+1)t^{n+1} j^{4n+1}_{l+\frac{1}{2},k} e^{-j^4_{l+\frac{1}{2},k}t} = \frac{n!}{6\pi}. \quad . \quad . \quad . \quad (34)$$

(31) and (32) are valid for all values of n, (33) and (34) for positive integral values. A simple special case of both (31) and (32) is

$$\lim_{t\to 0} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{(2l+1)t}{j_{l+\frac{1}{2},k}} e^{-j^2_{l+\frac{1}{2},k}t} = \frac{1}{3\pi}. \quad . \quad . \quad (35)$$

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XXXI. *The Thermodynamical Treatment of Diffusion Potentials.* *By* J. J. Hermans *and* L. J. Oosterhoff *.

Introduction.

1. In a recent paper Mr. Guggenheim [1] states: "although the thermodynamics of cells with liquid-liquid-junctions have been completely worked out and there is nothing essentially new to add to the subject, they still appear to be incompletely understood by certain experimenters in this field."

We cannot agree with this opinion, however, for the usual thermodynamical treatment consists of the application of formulæ which have been derived for systems in equilibrium. It seems questionable whether such a treatment can be applied to cells with liquid-liquid-junctions, in which diffusion, *i. e.*, an irreversible process takes place.

2. This objection has, of course, been discussed before. In the usual derivation of the expression for the e.m.f.

* Communicated by the Authors.

of a cell one considers the change of thermodynamic potential caused by the reversible passage of an electric current. Now it is not clear without more ado whether the passage through the diffusion layer can be regarded as reversible or not. In this connexion Guggenheim [2] observes : " It seems reasonable to suppose that the two processes (*i. e.*, diffusion and flow of current) are merely superposed, and that the one may be ignored when considering the other." We shall show, however, that the possibility of superposing these processes (which means the *reversibility* of the transport of matter by the current) does not imply that the processes are independent of each other, *e. g.*, with regard to heat-effects ; and this leads to the break-down of the thermodynamical derivation. This demonstration will be given by means of an analogy which exists between a cell with diffusion and a short-circuited cell, as

$$H_2/HCl\ c_1/AgCl,\ Ag,\ AgCl/HCl\ c_2/H_2. \quad . \quad . \quad (1)$$

A cell with diffusion is nothing but an *internally short-circuited* cell of this type.

3. It is remarkable that the thermodynamical treatment, in the case of ideal solutions, yields the same result as the kinetic theory, which makes use of the equations of motion for the ions, and therefore is in principle more justified. Probably this has led to the opinion that the same thermodynamical treatment can also be applied to non-ideal solutions. The purpose of this paper will be to prove, first, that this opinion is wrong, and secondly, why both treatments yield the same result in the special case of ideal solutions. From the fact that this thermodynamical treatment is found in several books on physical chemistry [2], it is clear that such a proof is not superfluous at all.

Derivation of the Thermodynamical Formula.

The formula, used by Guggenheim and others [3], generally is derived in the following way. Imagine a charge F, sent in a quasi-reversible way, *i. e.*, infinitely slowly, through a cell of the form

$$H_2/HCl\ c/HCl\ c+dc/H_2.$$

When this current enters the element on the left side, t^+ gram-ions of hydrogen are transported from the left- to the right-hand solution, t^- gram ions of chlorine from right to left, while 1 gram ion H is dissolved at the left-hand side and disappears out of the solution at the right-hand side (t=transport number).

The result is a decrease in thermodynamic potential

$$\mathrm{RT}(-t^+ d\ln a^+ + t^- d\ln a^- + d\ln a^+)$$

a being the activity.

With the aid of the relation $t^+ + t^- = 1$, this becomes

$$\mathrm{RT}t^-(d\ln a^+ + d\ln a^-) = \mathrm{RT}\,.\,2t^- d\ln a^{\pm},$$

where $a^{\pm}$ is the average activity of both ions.

On the other hand, the cell performed a work $\mathrm{F}\,d\mathrm{E}$; thus

$$d\mathrm{E} = 2\mathrm{RT}/\mathrm{F}\,.\,t^- d\ln a^{\pm}. \quad . \quad . \quad . \quad . \quad (2)$$

If there exists a finite difference of concentration between both solutions, one finds by integration

$$\mathrm{E} = \mathrm{RT}/\mathrm{F}\,.\int_{\mathrm{I}}^{\mathrm{II}} 2t^- d\ln a_{\pm}. \quad . \quad . \quad . \quad . \quad (3)$$

The potential difference between the solutions becomes

$$\mathrm{L} = -\mathrm{RT}/\mathrm{F}\,.\int_{\mathrm{I}}^{\mathrm{II}} (t^+ d\ln a^+ - t^- d\ln a^-). \quad . \quad . \quad (4)$$

Already, in 1927, Taylor [4] formulated an important inference from the formula (3): the e.m.f. of concentration cells with diffusion can never yield information about ion activities, because in (3) only the average activity appears.

As, however, the formula (3) will be proved to be wrong, this conclusion does not hold. In principle, therefore, the determination of ion activities cannot be regarded as impossible.

Criticism of the Thermodynamical Treatment.

1. *The Short-circuited Cell.*

Consider a cell

$$\mathrm{H_2/HCl}\,c_{\mathrm{I}}/\mathrm{HCl}\,c_{\mathrm{II}}/\mathrm{H_2}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For definiteness suppose $c_{\mathrm{I}} > c_{\mathrm{II}}$. The hydrogen-ions in the diffusion layer, moving from left to right, represent

a current i. The chlorine-ions, which move in the same direction, represent a current $-i$. This process can be considered as an internal short-circuit, as becomes quite clear when we compare the cell (5) with a cell of the form

$$\mathrm{H_2} \left| \begin{array}{c|c|c} \mathrm{HCl} & \mathrm{AgCl} \overset{r^-}{\sim\!\sim} \mathrm{AgCl} & \mathrm{HCl} \\ c_{\mathrm{I}} & \mathrm{H_2} \overset{r^+}{\sim\!\sim} \mathrm{H_2} & c_{\mathrm{II}} \end{array} \right| \mathrm{H_2}. \quad . \quad (6)$$

Here the diffusion layer is replaced by two junctions, the first consisting of two electrodes, reversible with regard to the negative ion and connected by a resistance r^-, the other consisting of two electrodes, reversible with regard to the positive ion and joined by a resistance r^+.

It is clear that each of these junctions can be regarded as permeable for one ion species only. They form a "diffusion layer" with separated motions of both ions.

It can easily be shown that the e.m.f. of this cell (6) agrees with the "thermodynamical" formula in case the resistances r^+ and r^- are constant. In that case (6) is analogous to a cell (5) with ideal solutions. If, however, r^+ and r^- depend on the current i, another value for the e.m.f. is found, and this case corresponds with the diffusion of non-ideal solutions.

In order to prove these statements, let us calculate the potential difference L between the solutions in (6).

Going from the left- to the right-hand solution through the anion-junction, we meet three potential differences. First we have to deal with the boundary HCl c_{I}/AgCl, then with the potential-difference ir^- in the resistance, and finally with the boundary AgCl/HCl c_{II}.

By choosing the resistances r^+ and r^- sufficiently large, the current i through them becomes so small, that one can use the thermodynamical formulæ for the potential differences between liquids and electrodes. Thus the "diffusion"-potential L becomes

$$\mathrm{L} = \mathrm{RT/F} \,.\, \ln a_{\mathrm{I}}^-/a_{\mathrm{II}}^- {}^+ - ir^-,$$

and, as the same considerations hold for the other junction, we have also

$$\mathrm{L} = -\mathrm{RT/F} \,.\, \ln a_{\mathrm{I}}^+/a_{\mathrm{II}}^+ + ir^+.$$

Leaving r^+ and r^- constant, one finds by elimination of i

$$L = \frac{\frac{1}{r^+}}{\frac{1}{r^+}+\frac{1}{r^-}} \frac{RT}{F} \ln \frac{a_I^+}{a_{II}^+} + \frac{\frac{1}{r^-}}{\frac{1}{r^+}+\frac{1}{r^-}} \frac{RT}{F} \ln \frac{a_I^-}{a_{II}^-},$$

which is exactly what would be arrived at by thermodynamical reasoning. Apparently the reciprocal resistances play the rôle of ion mobilities; the transport numbers (which here do not depend on the concentration) are

$$t^+ = \frac{\frac{1}{r^+}}{\frac{1}{r^+}+\frac{1}{r^-}}, \qquad t^- = \frac{\frac{1}{r^-}}{\frac{1}{r^+}+\frac{1}{r^-}}. \quad . \quad . \quad . \quad (7)$$

Thus L has entirely the same form as (4).

As soon, however, as we have to do with resistances that depend on i, *i. e.*, when ir^+ and ir^- are more general (non-linear) functions $F^+(i)$ and $F^-(i)$, the resulting potential difference L is no longer identical with that obtained by the usual thermodynamical treatment. In fact, this potential difference becomes

$$\left.\begin{aligned} L &= -RT/F \,.\ln a_I^+/a_{II}^+ + F^+(i), \\ L &= RT/F \,.\ln a_I^-/a_{II}^- - F^-(i), \end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

while the " thermodynamical " expression for all these cases remains

$$L_{th} = t^- \,.\, RT/F \,.\ln a_I^-/a_{II}^- - t^+ \,.\, RT/F \,.\ln a_I^+/a_{II}^+. \quad . \quad (9)$$

As soon will appear, these expressions are not identical. The difference between these values is exactly equal to the heat that is developed in the resistances if 1 Coulomb flows through the cell, diminished by the heat that would have been developed if only " diffusion " had taken place. (It is this extra heat, caused by the existence of an irreversible process, which makes the " thermodynamical " treatment yield a false result.)

One can see this as follows:—The heat developed in the resistances per unit time is $Q = i(F^+ + F^-)$. By applying an external e.m.f. we can make the potential difference between the two solutions $L + dE$. The current

through the resistance r^+ will now be $i+di^+$, and that through r^- : $i+di^-$. The total current through the cell is $di=di^+-di^-$.

The relations (8) become *

$$L+dE=-RT/F.\ln a_I^+/a_{II}^+ +F^+ +F^{+1}di^+,$$

$$L+dE=RT/F.\ln a_I^- a_{II}^+ -F^- -F^{-1}di^-.$$

Thus

$$-dE=-F^{+1}di^+=F^{-1}di^- \dagger.$$

Now, by definition,

$$t^+=di^+/di=\frac{F_-^1}{F_+^1+F_-^1},$$

$$t^-=-di^-/di=\frac{F_+^1}{F_+^1+F_-^1}.$$

The total heat developed per unit time in the resistances is

$$Q+dQ=i(F_+ +F_-)+di^+(F_+ +iF_+^1)+di^-(F_- +iF_-^1),$$

and thus the extra heat in consequence of the current

$$dQ=di^+(F_+ +iF_+^1)+di^-(F_- +iF_-^1).$$

During the time $1/di$, needed for one coulomb to pass through the cell, the extra heat becomes

$$dQ/di=t^+(F_+ +iF_+^1)-t^-(F_- +iF_-^1)=t^+F_+ -t^-F_-.$$

As has already been mentioned, this is exactly equal to the difference between the real L and L_{th}. By adding dQ/di to L_{th} (9), one finds in fact

$$t^-(RT/F.\ln a_I^-/a_{II}^- -F_-)+t^+(-RT/F.\ln a_I^+/a_{II}^+ +F_+)$$
$$=t^-L+t^+L=L.$$

Obviously the application of thermodynamics to the process considered as if we had to deal with a system in equilibrium, has led to the neglect of a finite amount of heat, developed in the "diffusion layer." This heat vanishes when the resistances r^- and r^- are constant. For in that case the transport numbers are given by (7), while F_+ and F_- are ir^+ and ir^- respectively. So it

* We here use the assumption that the two processes, diffusion, and flow of current through the diffusion layer, are superposed. Even in this case it will appear that these processes are not independent.

† F^1 stands for dF/di.

becomes clear why the usual application of thermodynamics was allowed in this case.

Criticism of the Thermodynamical Treatment.

2. *The Cell with Diffusion.*

Although the "thermodynamical" formula (9) is wrong, we may not yet conclude that it is now possible to determine ion activities with the aid of this cell (6). By elimination of L from the equations (8) one finds i as a function of $a^{\pm}$ and therefore also the e.m.f. of the cell, which is nothing but $F^{+}(i)$.

In the case of cells with *real* diffusion, *i. e.* (5), the relations are, however, still more complicated, because in that case the resistances depend on L itself. And therefore, eliminating L, one finds i as a function of the ion activities, which now cannot be written as a function of the mean activity only.

For ideal solutions the kinetic as well as the thermodynamical treatment yield

$$L=(t^{-}-t^{+})\,RT/F\,.\,\ln c_l/c_M.$$

The equations of motion which underlie this formula [5], are

$$\left.\begin{aligned}\rho^{+}\vec{v} &= -k\mathrm{T}\,\mathrm{grad}\,(\ln c)-\epsilon\,\mathrm{grad}\,\phi,\\ \rho^{-}\vec{v} &= -k\mathrm{T}\,\mathrm{grad}\,(\ln c)+\epsilon\,\mathrm{grad}\,\phi.\end{aligned}\right\}\quad .\ .\quad (10)$$

Here $\vec{v}$ is the average velocity of the ions, ϕ the electric potential, ρ^{+} and ρ^{-} are frictional factors.

From these equations it follows that, whenever a quantity of electricity passes infinitely slowly through the diffusion layer, no extra heat is generated. This was proved indeed in the above considerations, for (10) has essentially the same form as (8) with resistances r^{+} and r^{-} (here ρ^{+} and ρ^{-}) which do not depend on i. If, however, in the discussion of non-ideal solutions, one takes the point of view of the theory of Debye and Hückel, the equations of motion become [5],[6]

$$\left.\begin{aligned}\rho^{+}\Big(\vec{v}-\frac{\epsilon\kappa}{6\pi\eta}\,\mathrm{grad}\,\phi\Big) &= -k\mathrm{T}\,\mathrm{grad}\,(\ln a^{+})-\epsilon\,\mathrm{grad}\,\phi,\\ \rho^{-}\Big(\vec{v}+\frac{\epsilon\kappa}{6\pi\eta}\,\mathrm{grad}\,\phi\Big) &= -k\mathrm{T}\,\mathrm{grad}\,(\ln a^{-})+\epsilon\,\mathrm{grad}\,\phi.\end{aligned}\right\}\quad (11)$$

Here η is the coefficient of internal friction of the solvent and κ the reciprocal characteristic length from the theory of Debye and Hückel. From these equations it can be deduced that, when a quantity of electricity passes infinitely slowly through the diffusion layer, this involves a finite heat-effect, and in agreement with this the expression for L is different from (4). The left-hand side of (11) expresses that the resistances depend on grad ϕ. This is caused by the so-called electrophoretic effect. For the ions which transport the current di, it makes a great difference to find a potential gradient which existed beforehand. It is the influence of this potential gradient on the motion of the ions, which causes the finite amount of heat dQ/di.

Thus the determination of ion activities, *at least in principle*, cannot be regarded as impossible, and they are not "physically undefined" as Guggenheim supposes. All determinations of activities, which are based on pure thermodynamical formulæ, indeed only yield information about mean activities. For kinetic methods, however, this does not hold.

One can agree with Guggenheim that the measurements of Szabó [7], mentioned by him, do not allow any positive conclusion in the sense proposed by Szabó himself, as was already amply demonstrated by one of us [5]. On the other hand we should like to call attention to the work of Quintin [8], and also to an interesting attempt to obtain ion activities independently of diffusion potentials by means of a method, recently indicated by Szabó [9], involving measurements of non-isothermic cells.

McInnes and his collaborators [10] have used formula (3) for the calculation of mean activities from measurements of the e.m.f. of concentration cells with diffusion. From what has been said in this paper, it is clear that no value can be attached to this method.

It might seem that their results confirm the formula (3). In this respect we may, however, cite L. Ebert (*Handb. d. Exp. Physik*, xii. 1, p. 332 (1932)), who makes the following remark in connexion with this method: "Weiter erfordert die Abschätzung der Fehler, welche von den graphischen und rechnerischen Auswertung herrühren, Aufmerksamkeit. Je nach der gewählten speziellen Interpolation der Ionenaktivitäten (he means the average activity) können aus einer Messungsreihe recht

verschiedene Kurven von Überführungszahlen errechnet werden."

If this were true, it might not be difficult to obtain the right transport number from the e.m.f.'s concerned; and so the formula (3) could not be regarded as definitely confirmed.

We are very grateful to Prof. van Arkel for his keen interest in this work, which furnished the occasion for many improvements in the exposition.

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XXXII. *Mercury Arcs in Capillary Tubes.* *By* Dr. V. Voss, *University of Pretoria* *.

IN a previous paper † a simple non-oscillating mercury arc was described. This consisted of a horizontal capillary tube of pyrex or silica, to the ends of which wider vertical tubes were sealed. Mercury was introduced, and then corks, through which iron wires were passed,

* Communicated by the Author.
† Phil. Mag. vol. xx. July 1935, p. 179.

were pushed down the vertical tubes till they touched the mercury surfaces. Sealing-wax was run in above the corks. A small direct current from a 212 volt generator was passed through this tube, and the mercury thread in the capillary was split with a pointed flame. The arc would then burn quite steadily (*i. e.*, without the mercury columns oscillating) and with great brilliance. It was frequently noticed that the final value of the current seemed to be independent of the external resistance in the circuit provided this was not too high. Thus with a tube of inner radius 0·37 mm. the final current was always 0·06 amp., with another of radius 0·54 mm. it was 0·074 amp., while with a third of radius 0·99 mm. it was 0·134 amp.

If, however, the external resistance was reduced too far the arc would go out before the final steady current was reached.

It seemed worth while investigating this, and sets of readings were taken with various tubes, the following being a typical set. The internal radius of the tube

c.	*i.*	l.	r/l.	v/l.	w/l.
0·30	0·134	1·58	554	74	9·9
0·28	0·134	1·52	543	73	9·8
0·27	0·134	1·50	531	71	9·5
0·26	0·134	1·46	525	70	9·4
0·25	0·135	1·39	519	70	9·5
0·24	0·14	1·33	475	66	9·3
(later)	0·14	1·27	498	70	9·8
0·23	0·142	1 21	472	67	9·5
(later)	0·141	1·19	488	69	9·7
0·22 here the gap gradually closed up completely.					

The third figures in the second column are estimated.

was 0·99 mm. and its external radius 3·56 mm. The inner radius of the vertical side-tubes was about 3 mm. c is the current before the arc was struck, i the current through the arc, l the length in cm. of the arc gap, r/l the resistance per cm. of the arc (ohms per cm.), v/l the average potential gradient (volts per cm.) along the arc, and w/l the watts per cm. generated in the arc. c and i were read on an ammeter, and, knowing the voltage of the generator, r, v, and w could be calculated. The gap length was read (through a dark screen) on the

eyepiece scale of a short-focus telescope, and converted to cm.

When the gap is formed the mercury columns are pushed apart. The corks above the mercury in the vertical tubes are slightly porous, and so they and the air in them are compressed. This opposing pressure will alter only slightly for small changes in l because the vertical tubes are so much wider than the capillary tube. The joule-heating in the gap raises the temperature, and the gap very slowly lengthens. The temperature gradient in the walls of the capillary increases, and so does the area from which heat is being lost to the surroundings. For both these reasons the rate at which heat is lost to the surroundings increases, and when this rate is equal to the rate of production of heat the emperature becomes constant, and so the resistance of the arc and the current become constant, and equilibrium sets in. If the external resistance is decreased the joule-heating in the gap increases; but if the pressure remains approximately constant so does the temperature of the vapour, and the gap length increases, the radiating surface (and also the resistance of the arc) therefore increasing in the same proportion till again heat is lost at the same rate as it is produced. And so the watts generated per cm. of gap will remain approximately constant. Since $w=i^2r$, and w/l and r/l are constant, i will be constant, as is found to be the case.

Suppose c is constant, *i. e.*, the external resistance R is fixed. Then

$$r=212\left(\frac{1}{i}-\frac{1}{c}\right);$$

$$\therefore \quad v=ir=212\left(1-\frac{i}{c}\right),$$

$$\therefore \quad w=iv=\frac{212(c-i)i}{c}=ki(c-i).$$

$$\therefore \quad \frac{dw}{di}=k(c-2i). \qquad \ldots\ldots\ldots \quad (1)$$

If $c>2i$, then a decrease in i means a decrease in w, while if $c<2i$, a decrease in i means an increase in w.

Suppose the final arc current i is constant, although c is altered, as in the first few readings above.

$$w = iv = \frac{212i(c-i)}{c};$$

$$\therefore \quad \frac{dw}{dc} = \frac{212i^2}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is always positive.

Thus as the initial current c is raised by decreasing the external resistance the watts consumed in the arc increase.

When a mercury arc is struck in a large evacuated space a rise in temperature will at first cause a lowering of resistance because of the increase in the vapour pressure and the consequent increase in the amount of ionization; but when an arc is formed at high pressure in the tiny space in a capillary tube a rise in temperature will produce a rise in the resistance, because the increase in vapour density restricts the free paths and so the ionizing powers of the electrons.

Now consider the table of readings above. When c is 0·30 amp., after the arc is started the current gradually decreases to its final steady value 0·134 amp. (it fluctuates slightly about this value), and the gap increases to a definite maximum length 1·58 cm. At first $c<2i$, and as i decreases w increases to a maximum when $c=2i$. After this w gradually decreases. Suppose after i has decreased to 0·134 there is a momentary cooling of the surface of the tube (due say to a draught). The arc cools, its resistance decreases, i increases and w increases. Thus the temperature is once more raised, *i. e.*, the conditions are inherently stable. But suppose c is say 0·24 amp. and is always less than $2i$. When the gap is formed the rate of heating is greater than the rate of cooling, and the temperature rises until equilibrium is again attained; but now if there is a momentary cooling of the arc the current rises and w decreases. So here the equilibrium is unstable, but the changes are very slow, and the gap, which is always moving towards the positive (hotter) end of the arc, may have moved right across to the end of the tube before any large changes in current or gap length occur. The following is one of an actual set of readings :—

c.	*i*.	Eyepiece divisions.
amp.	amp.	
0·24	0·173	1·45
	0·146	1·97
	0·140	2·04
	0·137	2·08
	0·139	2·08
	0·142	2·02
	0·143	2·00

The above series took 25 minutes.

The smaller c is made the more rapidly does w decrease (eq. 2), and when c is too small w may be too small to maintain the gap after it has been formed with a pointed flame ; so the gap cools, i gets greater, and therefore w gets still less, till the gap closes altogether. In the table quoted this happens when $c \leqslant 0{\cdot}22$ amp.

It was found that when the external resistance was made too small, and c therefore too great, that after the arc was started the current would decrease, but before it became steady the arc would suddenly go out. With the tube for which the above readings were given this happened when c was greater than 0·41 amp. It will be seen from the results given that as c increases the values of watts per cm. slowly increase. Possibly for larger values of c the gap may not be able to increase sufficiently, owing to the gradually increasing opposing pressure, to allow watts per cm. to remain even approximately constant. The temperature of the arc space will then go up, and it may be that the increasing density of the vapour may restrict the movement of the electrons to such an extent that they cannot acquire sufficient velocity to produce ionizations by collision.

The sealing-wax at one end of this tube was now softened and the cork pushed in a bit further. The following results were now obtained :—

c.	*i*.	l.	*r*/l.	*v*/l.	*w*/l.
amp.	amp.	cm.			
0·41	0·136	1·61	647	88	12·0
0·38	0·136	1·57	638	87	11·8
0·32	0·136	1·42	631	86	11·7
0·28	0·137	1·28	617	85	11·6

The arc was obviously much brighter than before, and as can be seen the watts per cm. were higher, and so the temperature of the arc space was greater; but to offset this the potential gradient was also greater. The values of i for the larger values of c fluctuated considerably more than before: thus for $c=0{\cdot}41$ i varied between 0·131 and 0·140, but 0·136 was about the average. Another tube having roughly the same bore as the above was now taken, but instead of sealing up the side-tubes with corks and sealing-wax close-fitting iron cylinders were pushed down to on the mercury and sealing-wax run in above these. Iron wires led out through the wax. Here there was less chance of appreciable air cushions remaining above the mercury. The following results were obtained: —

c.	i.	l.	r/l.	v/l.	w/l.
amp.	amp.	cm.			
0·26	0·095	0·67	2113	201	19·1
0·232	0·097	0·66	1926	187	18·1
0·20	0·100	0·63	1683	168	16·8
0·18	0·102	0·55	1636	167	17·0
0·16 gap gradually closed up.					

Here for the larger values of joule-heating (larger values of c) the mercury columns were pushed apart against a progressively increasing pressure. Thus instead of the temperature remaining constant and the radiating surface increasing, as previously, the latter increased to a lesser extent, but the temperature went up until equilibrium was reached. So the watts per cm. went up appreciably, and the final values of the arc currents became progressively smaller.

For the higher values of c the path of the discharge was tortuous, very like that seen in a long discharge-tube when the pressure is reduced to the stage where the discharge first passes. As c was decreased the path became more diffuse, just as in the discharge-tube when the pressure is further lowered, and, no doubt, for the same reason.

In the previous paper* mention was made of a flickering in the light intensity from these non-oscillating arcs. This point has recently been investigated again and the flickering traced to a voltage fluctuation originating

* *Loc. cit.*

in the generator. The flicker frequency was found, with the stroboscopic disk, to be 96·4 per second. This was tried for various arcs of different bores, but the frequency was always the same. When current from the same generator was passed through a neon or a helium discharge-tube no sign of flickering was at first observed; but when a high resistance was included in the circuit and the discharge-tube connected across a variable portion of this resistance a point was reached, just before the discharge across the tube ceased, when the intensity of the discharge began fluctuating, and the frequency of this flicker was found by the stroboscopic disk to be the same as that in the mercury arcs, viz., 96·4 per sec. The generator has 91 segments and 4 brushes; it makes 1443 revs. per minute. If in each revolution one segment, or a few adjacent segments, make poor contact with each brush (this might, *e. g.*, be due to dirty segments or a wobble in the commutator shaft) a dip in voltage will occur, and the frequency of this dip will be $\frac{4\times1443}{60}=96{\cdot}2$ per second.

This was repeated with a 600 volt generator, with similar close agreement.

In the case of discharge-tubes the flicker only becomes apparent when the voltage decreases nearly to that critical value below which the discharge will not pass; but always in the case of the arc the greater the voltage across the arc the more conspicuous is the flickering in intensity. For these higher voltages the final steady current through the arc is practically constant, and so the voltage across the arc is proportional to the resistance of the arc. The voltage across the arc $v=\mathrm{V}-i\mathrm{R}$, where V is the voltage of the generator, i the current through the arc, and R the external resistance. Thus

$$v=\mathrm{V}-\frac{\mathrm{V}}{\mathrm{R}+r}.\,\mathrm{R}=\mathrm{V}\frac{r}{\mathrm{R}+r},$$

r being the resistance of the arc. A change in V does not produce an immediate change in r. So, regarding r as constant, we have $\frac{dv}{d\mathrm{V}}=\frac{r}{\mathrm{R}+r}$, and we see that the larger r is (and therefore the voltage across the arc) the greater is the change in the voltage across the arc produced by a fluctuation in the voltage of the generator.

XXXIII. *Variation of Resistance and Power Factor of Glass Condensers with Radio-frequencies.* *By* HAZEL M. FLETCHER, *Kansas State College, Manhattan, Kansas* *.

ABSTRACT.

THE resistance and power factor of glass condensers were measured at radio-frequencies of 76 to 8250 kilocycles. The resistance-frequency curves show small maxima between 1300 and 1900 kilocycles, and the power factor-frequency curves have an irregular flat region at the same frequencies. The power factor decreases from 60 to 70 per cent. in the frequency range. The resistance was found to vary inversely as the frequency and inversely as the capacity. The greater portion of the resistance of glass condensers as well as of air condensers is due to dielectric losses. In the case of glass condensers a portion of the dielectric absorption is due to polar orientation as explained by the Debye theory.

THE physicist and chemist have devoted considerable attention to the behaviour of dielectrics under alternating stress to study the mechanisms involved in dielectric absorption and dielectric loss. In many insulating materials this loss may be accounted for by ions or free electrons present which give rise to conduction currents. In other insulating materials the loss can be attributed to the dielectric polarization and orientation of the neutral molecules when there are no free electrons or ions. The first explanation is supported by the inhomogeneity theory of Maxwell and Wagner [1], and the second by the Debye [2] theory of orientation of polar molecules. The variation of the dielectric constant and power factor with temperature and frequency is one of the methods used to ascertain the sizes, shapes, and structure of the molecules.

Many dielectrics are under investigation to discover anomalous dispersion whose existence in the audio- and radio-frequency region explains the complicated behaviour of many insulators. Much study is being devoted to this phenomenon in certain crystalline substances and binary mixtures. Probably most, but not

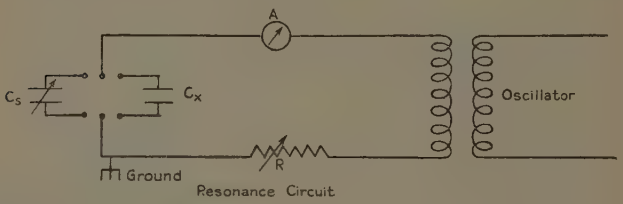
* Communicated by the Author.

all, workers [3], [4], [5], [6], [7], [8], [9] believe the maxima found in the dielectric constant-frequency and power factor-frequency curves are best explained by the molecular orientation theory of Debye. Sommerman [10] reported a small power factor maximum in paraffin which he believes due to orientation of polar aggregates of large radius.

Glass is a substance whose electrical properties and dielectric constant vary with a change in its composition. It is an excellent dielectric at ordinary temperature; at all temperatures it is an electrolytic conductor. A film of moisture condensed on the surface of the glass causes a part of the electrical conductivity.

Because of the fact that the experimental results of glass [11] are not satisfactory, it was thought that it might

Fig. 1.



be worth while to investigate the variation of resistance and power factor of glass condensers with frequency. It has been reported that the power factor of some glasses increases with frequency and of other glasses decreases. There may be inversion points on the power factor-frequency curve, but this has not been proved.

Apparatus and Method of Measurement.

For making the resistance measurements the resonance-substitution method was used. The diagram of the apparatus is shown in fig. 1. C_s is a variable air condenser which was calibrated against a standard condenser. The oscillator, which consisted of a Hartley series vacuum-tube circuit with a variable air condenser and various sizes of inductance coils, was calibrated by means of overtones with an oscillator which had been previously calibrated. The condenser, C_x, under test was put in the circuit and resonance obtained by tuning the oscillatory circuit. The standard low-loss condenser, C_s, was then put in the

circuit and tuned to resonance. The amount of resistance added to give the same reading of the ammeter is the difference in resistance of C_x and C_s. This difference was not taken as the resistance of C_x.

The standard condenser, C_s, was found to have the same resistance as the thick-plate aluminium condenser No. 2, whose resistance the author [12] had measured by means of the calorimeter method as described by Ramsey [13] The resistance of this condenser is ·042 ohms at 170 metres wave-length, and its capacity is ·000180 μf. The resistance of the standard condenser at any wave-length and capacity was deduced by the relation

$$R_s = \cdot 042(\lambda_s/170)^{\frac{1}{2}}(\cdot 000180/C_s)^{\frac{3}{2}}$$

which the aluminium condenser was found to obey. The resistance of C_x is the sum of R_s and the resistance which was added when C_s was in the circuit. However, R_s was small, and the error in resistance measurement in omitting R_s would usually be of the order of one per cent.

Eight glass plate condensers were used in this investigation. They were made of photographic glass plates about 12×10 cm. in area and 1·5 mm. thickness and metal plates about ·5 mm. thick. The approximate percentages of the main constituents in the glass are: SiO_2, 72 per cent.; CaO, 8·5 per cent.; MgO, 4·5 per cent.; Na_2O, 13 per cent. The metal plates were of sheet copper except those in two of the condensers. One condenser had aluminium plates and the other zinc. These condensers varied in capacity from ·000113 to ·000560 microfarads. The measurements were taken in the frequency range of 76 to 8250 kilocycles per second.

Results.

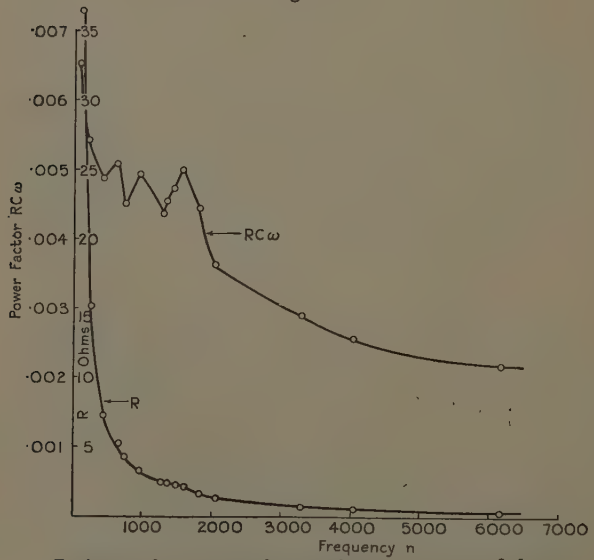
The accompanying table shows the data of equivalent series capacity (C), equivalent series resistance (R), and power factor ($RC\omega$) for different frequencies (n) for one of the condensers. A similar table could be given for seven condensers ranging from ·000113 μf. to ·000505 μf. In fig. 2. are shown the resistance-frequency and power factor-frequency curves for the same condenser. The resistance-frequency curve shows a small maximum at 1500 kilocycles, and the power factor-frequency curve has an irregular flat region in the same range. These are typical curves, for all the condensers behaved in a like manner

except the one with the zinc plates, which was very erratic. The resistance maxima of the other condensers came at 1600, 1200, 1900, 1300, 1900, and 1500 kilocycles. These maxima can be attributed to the orientation of a polar

C.	n.	R.	RCω.
·000228 μf.	6200×10^3	·24 ohms	·00213
·000228	4050	·44	·00255
·000228	3300	·61	·00288
·000230	2080	1·20	·00361
·000228	1830	1·70	·00445
·000240	1600	2·08	·00502
·000229	1510	2·18	·00474
·000230	1350	2·33	·00454
·000228	1320	2·32	·00439
·000240	980	3·33	·00493
·000238	655	5·14	·00510
·000230	465	7·25	·00487
·000230	250	15·0	·00541
·000242	118	36·4	·00654

Dielectric constant : 5·33.
Index of refraction : 1·51.

Fig. 2.

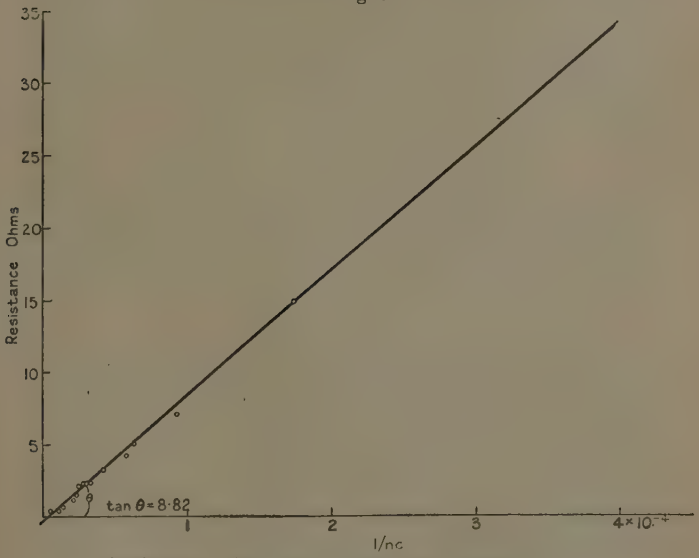


Resistance-frequency and power-frequency curves of glass plate condenser.

aggregate of atoms in the molecule. The power factor-frequency curves show, except in the region of the maxima, a decrease of power factor with frequency. There is a 60 to 70 per cent. decrease in the range of the radio-frequencies used.

When logarithms of resistance were plotted against logarithms of frequency the curves were straight lines for each condenser with slope of negative one, which showed that the resistance varied inversely as the fre-

Fig. 3



Resistance—$1/nC$ curve of glass plate condenser period.

quency. When logarithms of capacity were plotted against logarithms of resistance the curves approximated a straight line with slopes of negative one, thus showing that the resistance varied inversely as the capacity. Fig. 3 shows the curve of resistance and $10^{-4}/nC$ for the condenser, whose data are given in the preceding table, when n is frequency per second and C is capacity in farads. Only the last point (36·4 ohms) came off the straight line. For the curves of the other condensers the corresponding point came on the straight line for two and not for the other four. The slope of this curve is 8·82. The resistance

of this condenser for any capacity and frequency can be calculated by the equation

$$R_x = 8{\cdot}82(10^{-4}/C_x n_x).$$

The slopes of the same curves of the other condensers are 8·26, 9·33, 6·25, 8·00, 8·08, and 8·46.

Ramsey [14] and Morris [15] found that the resistance of air condensers varied directly as the wave-length and inversely as the three halves power of the capacity. Dye[16], Moullin[17], Wilmotte[18], and Jackson[19] found relations in which the resistance varied inversely as the frequency and inversely as the square of the capacity. Due to the fact that the resistance of air condensers and glass condensers varies inversely as the frequency, the greater portion of the resistance of both is due to dielectric losses.

Acknowledgments.

It is a pleasure to express my appreciation to the Department of Physics of Indiana University for the use of apparatus placed at my disposal while I was a guest in the department. I also wish to thank Dr. R. R. Ramsey for his assistance and criticisms.

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XXXIV. *Supplementary Note on "A General Proof of Saint-Venant's Principle"* *. *By* J. N. Goodier, *M.A., Ph.D.* †

IN the paper in question it is pointed out that Saint-Venant's Principle follows from the conservation of energy in any solid which is deformed elastically or plastically to a definite state of strain. The extent of the region of the solid influenced by a self-equilibrating distribution of force on a small area of the surface is estimated for the elastic state. The method employed, however, involves unessential implications which are open to objection, and the following is proposed to replace it.

Let the solid be loaded on a small area whose linear dimensions are of order a, with tractions which combine to give zero resultant force and couple. The tractions are of order p, and the slopes of the stress-strain curves of the material are of order E.

Let one element of the loaded area be regarded as fixed both in position and orientation. Then since the strain must be of order p/E, the displacements of points within the area are of order $p/\mathrm{E} \cdot a$. The work done by the traction on an element $d\mathrm{A}$ is of order $p/\mathrm{E} \cdot a \cdot pd\mathrm{A}$. To obtain the order of magnitude of the total work done by the applied tractions, it is sufficient to regard all such contributions as positive. The total work is thus of order $p/\mathrm{E} \cdot a \cdot pa^2$ or p^2a^3/E.

As before, since stress of order p implies strain-energy of order p^2/E *per unit volume*, the region within which the stress is of order p must have a volume comparable with a^3, and, therefore, the influence of the tractions cannot be appreciable at distances from the loaded area which are large compared with its linear dimenisons.

Ontario Research Foundation,
April 16, 1937.

XXXV. *Notices respecting New Books.*

The Theory of Metals. By A. H. Wilson. [Pp. viii+272.] (Cambridge University Press, 1936. Price 18*s.*)

THE appearance of a book by Wilson on the theory of metals is particularly welcome in view of his own considerable

* Phil. Mag. ser. 7, vol. xxiii. p. 607 (April 1937).
† Communicated by theAuthor.

contributions to the subject. As would be expected in a monograph based on an Adams prize essay, mathematics is prominent, and much of it is somewhat formidable ; but the " commentary " and general discussions are admirably clear and straightforward.

An historical introduction gives a critical summary of the classical treatment of conductivity by Drude, Lorentz, and others, and describes Sommerfeld's application of Fermi-Dirac statistics. The general theory of the motion of electrons in a perfect lattice is then developed, and applications of the theory to equilibrium properties are made. There is a full description of the zone structures of metals belonging to the different crystal classes, and methods of approximating to the wave functions for the electrons are developed, leading up to a discussion of cohesion. A section on magnetic properties is notable for a powerful attack on the difficult problem of metallic diamagnetism. A chapter on optical properties includes discussions on dispersion, X-ray emission and absorption spectra, and the photo-electric effect.

The treatment of conductivity is rather sharply divided into two parts, with considerable advantage. The first deals with the formal theory, in which, on the basis of the postulate that a free path may be defined, a derivation is given of the equations for conductivity and for the various magneto-electric and thermo-electric effects. The theory of the mechanism of conductivity, essentially of the interaction between electrons and lattice vibrations, is then developed, leading to results which can, in part, be at once incorporated in the formal scheme. As Wilson remarks : " It is perhaps unfortunate that so much attention has been paid to the resistance of metals, since it is probably one of the least characteristic properties of the substance, and depends on the electronic distribution and the elastic constants in a very complicated way " ; but in the attack on the problem he evades none of the difficulties.

The final chapter summarizes the position in connexion with superconductivity, and there are useful appendices on Fermi-Dirac statistics and on surface phenomena.

The essential purpose of the book is to develop the theory as rigorously as possible. " It would indeed be easy," says the author, " to induce a spurious idea of the successes of the theory by suitable manipulation of arbitrary constants, but much more numerical work . . . is necessary before we can be really satisfied with the quantitative side of the subject." There is, in fact, little detailed consideration of actual metals with a view to confronting theoretical conclusions with experimental results. This is consistent with the general plan of the book, although many readers, while admiring the

skill with which the theoretical flight is conducted, will possibly feel the desirability of more frequent landings.

The author has, however, set himself a particular task, that of dealing rigorously and more or less completely with the theory, and this he has carried out with conspicuous success. The book is quite distinct in character and achievement from any other which has appeared on the subject, and it should be in the hands of all those who are seriously interested in the difficult but fascinating study of the theory of metals.

The Newer Alchemy. By Lord Rutherford. [Pp. 67 with 13 plates and 6 figures.] (Cambridge University Press. Price 3*s.* 6*d.*)

Good wine needs no bush. This charming little book will surely no sooner be heard of than bought by all interested in the new and exciting subject with which it deals.

Based on the Henry Sidgwick Memorial Lecture delivered at Newnham in 1936, Lord Rutherford has produced in eminently readable form a pleasantly coherent account of "the remarkable increase in our knowledge . . . of the problem of the transmutation of the elements . . . that has come so rapidly in the last few years."

The production and use of the various bombarding particles in artificial transformation are lightly, though adequately, outlined. Recent discoveries of new elements, such as radio-bismuth and radio-sodium, are touched on, and general conclusions are drawn in regard to future trends.

No one generally interested in this new and rapidly expanding branch of knowledge can afford not to read this enthusiastically written and stimulating book.

Atomic Spectra and the Vector Model. By A. C. Candler. Vol. I. *Series Spectra.* [Pp. viii+238.] Vol. II. *Complex Spectra.* [Pp. vi+280.] (Cambridge University Press, 1937. Price 15*s.* each vol.)

There are sections on atomic spectra in most of the books dealing with the quantum theory at various stages of its development; usually, however, a comparatively small number of spectra are considered, the primary purpose being simply to illustrate applications of the theory. There have been few books aiming at a detailed descriptive survey of the spectra themselves. In these volumes a complete survey is given of the spectra of the elements, together with accounts of related topics, use being made throughout of the consistent scheme of nomenclature which is now firmly established.

There is little doubt that the most effective method of obtaining a coordinated group of the vast range of observational

material is by making use of the vector model so characteristic of the first phase of the quantum theory. The author has practically excluded wave mechanical considerations, except for statements, where desirable, of the conclusions to which they have led. The *deux ex machina* effect may be deplored, but it must be admitted that for the kind of survey made here detailed quantum mechanical discussions would have been out of place.

There is no sharp sub-division of subject-matter between the two volumes, which have a common index. In the first volume the simpler series spectra—of hydrogen, the alkalis, and the alkaline earths—are described and discussed, and there are accounts of the Zeeman and Stark effects, and an excellent chapter on the periodic system. In the second volume two-electron atoms are considered further, and the more complex spectra are treated in chapters on the elements of the short and long periods and the rare earths. Other chapters deal with intensity relations, sum rules, hyperfine structure, quadripole radiation, and fluorescent crystals.

Throughout there is a wealth of illustrative data in tabular form, and there are numerous admirably thought-out diagrams, as well as eight excellent plates. Appendices to the first volume include a list of physical constants (here there are some departures from the usually adopted values, the reasons for which should have been given) and a Rydberg term table ; the second volume has an eighteen page bibliography of spectra hyperfine structure, arranged in alphabetical order of the chemical symbols of the elements.

The author has properly made use of previous general works and compilations (such sources are fully acknowledged) as well as of original papers. In spite of the mass of material which he has had to organize he has maintained a fresh and fluent style of presentation, and above all remarkable clearness. The book will undoubtedly be of value to the specialist ; but at the same time it will be suitable for those who have little or no previous knowledge of spectra. The survey which it provides of the whole subject of atomic spectra, while comprehensive, is manageable, and the author appears to have been most successful in his aim of presenting the arguments in such a way as to be " within the easy comprehension of every working physicist." The book is a valuable addition to the literature of spectra, and the thanks of physicists are due to the author for undertaking so exacting a task, as well as their congratulations on the admirable manner in which he has carried it to completion.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]